The course follows the book *Differential Equations with Boundary-Value Problems* by D. G. Zill, and W. S. Wright, 8th Ed, Cengage Learning, 2012

The other usefull sources, which were helpful to prepare my notes:

- *Ordinary Differential Equations* (Dover Books on Mathematics) by Morris Tenenbaum, Harry Pollard
- James Stewart's Calculus
- Paul's Online Math Notes
- Dr. Howell's Lecture Notes
- S. O. S. Mathematics

By *Miroslav Stibor*, Zaman University.

You can get all the below chapters in one PDF (5 MB): Differential equations.pdf

List of chapters

The Laplace Transform

Fourier series

Partial differential equations, boundary value problems

Introduction to differential equations

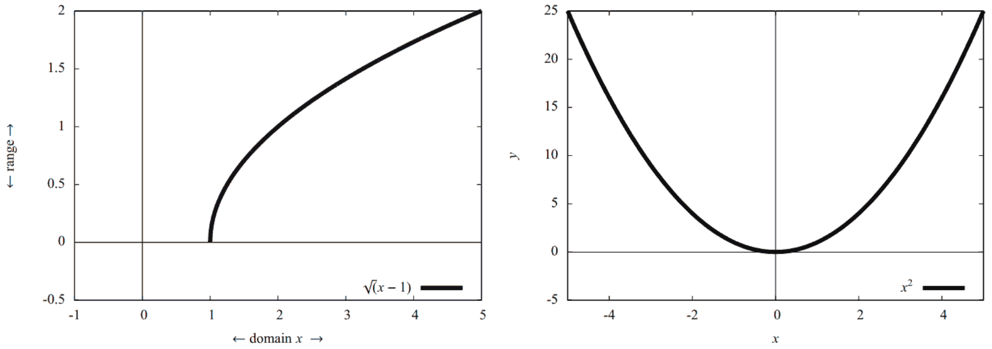
The importance of DE (motivation)

Differential equations are equations which express a relationship between variables and their derivatives. Typically, when variable changes during time and other variable is depending. Differential equations play important role in describing problems e.g. from

- physics,
- dynamics,
- medicine,
- structural analysis,
- populations studies.

What is a function?

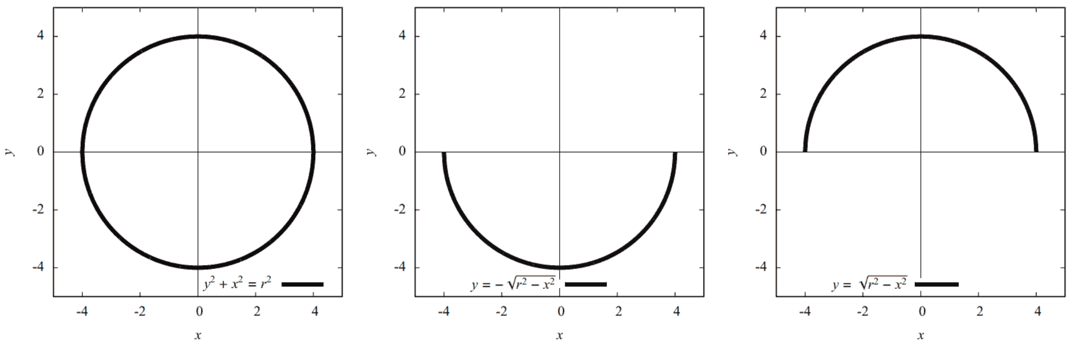
1. Function has a domain/range.
2. For each value from domain it has one function value.



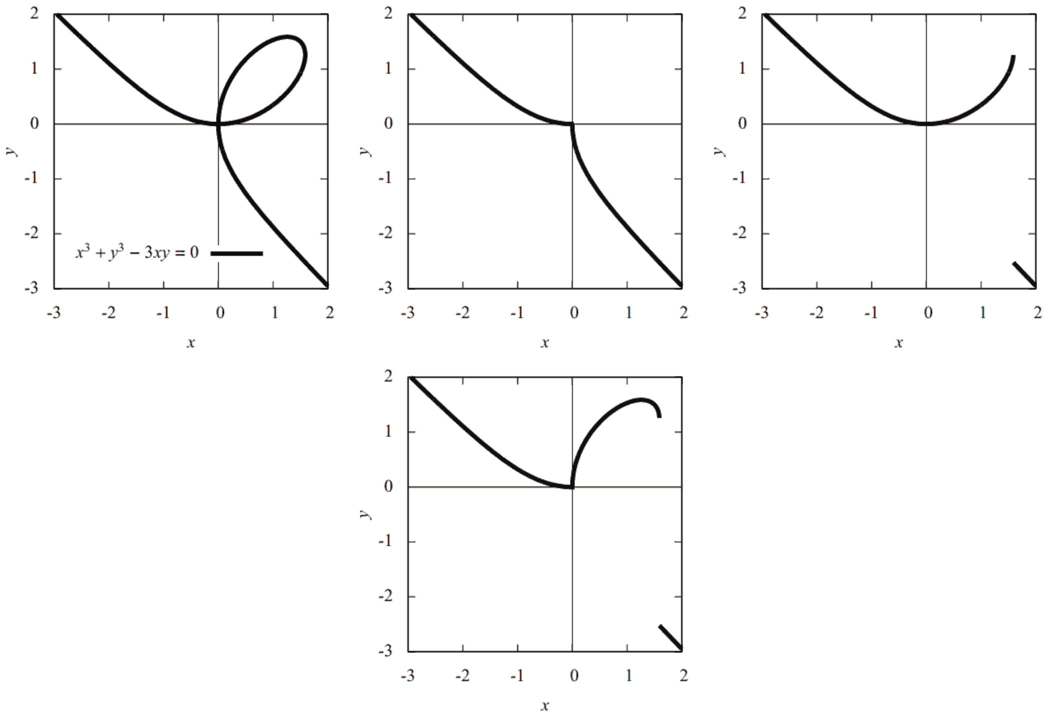
Left graph: *$y = \sqrt{x-1}$, $x \geq 1$ is a function, since it has said domain and it assigns for each value from domain one function value.*
Right graph: *$y = x^2$ is not a function. But because we assume that domain is either $(-\infty, \infty)$ or all values which make sense, it is a function also then.*

Implicit function

Implicit function is expressed as $f(x, y) = 0$ and **x and y are not dependent variables. Such functions are not solved easily for y in terms of x,** so they are unwelcome.



Implicit function $x^2 + y^2 = r^2$ can be rewritten to its explicit form either as $y = \sqrt{r^2 - x^2}$ or $y = -\sqrt{r^2 - x^2}$.



To draw implicit function is a tedious task. Since to one value of x may belong more than one function value, usually an iterative process is needed. The implicit function $f(x,y) = x^3 + y^3 - 3xy$ depicted above can be converted (with a loss of information) to explicit form, which is easy to manipulate. We can find million ways how to convert, only some of them are shown.

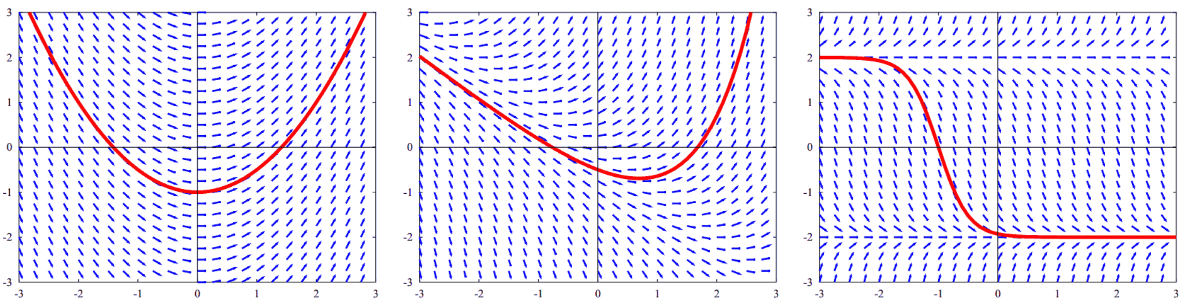
A note about differentiating implicit function. To express $y' = dy/dx$ chain rule has to be applied (in such case y become dependent on x):

$$\begin{aligned} x^3 + y^3 - 3xy &= 0 \\ 3x^2 + 3y^2y' - (3xy' + 3y) &= 0 \\ y'(3y^2 - 3x) &= 3y - 3x^2 \\ \frac{dy}{dx} &= \frac{y - x^2}{y^2 - x}, \qquad y^2 \neq x \end{aligned}$$

Ordinary differential equations (ODE)

It is important to keep in mind that once we see y', its meaning is a tangent at given point. As k is tangent in $y = kx \implies k = y/x$, so $y' = dy/dx$.

If we are given simple differential equations $y' = x$, we have enough information to draw a tangent at each point. If we make a grid of points, we can draw small tangent at each grid point, as in the graphs shown below. Then we have graph of differential equations which is called direction field. These tangents are guides, they show where the solution might pass.



Differential equations drawn as a direction field for three different DE. Also one particular solution drawn in red color for each.
From left: $y' = x$, $y' = x + y$, $y' = (y+2)(y-2)$
When constructing direction by hand, it is efficient to find isoclines—lines connecting grid points with the same slope (these can be observed in the second example).

Verification of explicit solution

Example

Verify that $y = e^{-x}$ is a solution of DE $y' + y = 0$.

We have a proposed solution of the differential equation. We mean, that if we substitute proposed solution into $y,\ y'$ within DE, the equality have to hold.

If $y = e^{-x}$, then it $y' = -e^{-x}$. Let us substitute $y,\ y'$ into DE to check, whether equality holds:

$$\begin{aligned} -e^{-x} + e^{-x} &= 0 \qquad x \in \mathbb{R} \\ 0 &= 0 \end{aligned}$$

It is shown above that $y = e^{-x}$ is a solution of DE $y' + y = 0$

Example

Verify that $y = e^{x}$ is a solution of DE $y' = e^{x}$.

From the proposed solution $y = e^{x} \implies y' = e^{x}$. So it is obvious that $y = e^{x}$ is a solution of given DE, $x \in \mathbb{R}$.

Example

Verify that $y = x \arcsin x + \sqrt{1 - x^2}$ is a solution of DE $d^2y/dx^2 = 1/\sqrt{1 - x^2}$.

First, it can be observed, that $1 - x^2$ have to be greater than zero or

$$-1 \leq x \leq 1$$

Now the domain is valid also for $\arcsin x$. Note that d^2y/dx^2 is an other notation of writing y''. We have to differentiate proposed solution two times, than we can substitute into DE.

$$\begin{aligned} y' &= \arcsin x + \frac{x}{\sqrt{1 - x^2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{1 - x^2}}(-2x) = \arcsin x \\ y'' &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

It is shown above that $y = x \arcsin x + \sqrt{1 - x^2}$ is a solution of DE $d^2y/dx^2 = 1/\sqrt{1 - x^2}$.

Note: The first derivative is somehow complicated and one might have doubts whether y' has been evaluated correctly. In the case of doubts, the numerical check can be done by calculator from random two points which are close to each other. The slope between these points has to be in chord with $y'(x)$.

Example

Verify that $y = x^2$ is a solution of DE $xy' = 2y$.

$$y = x^2 \implies y' = 2x$$

Now substitute into DE:

$$\begin{aligned} x \cdot 2x &= 2y \qquad x \in \mathbb{R} \\ 2x^2 &= 2y \\ x^2 &= y \end{aligned}$$

It was shown above that $y = x^2$ is a solution of the DE.

Example

Verify that $(1 + x^2)y' = xy$ has a solution $y = \sqrt{1 + x^2}$.

$$y = \sqrt{1 + x^2} \implies y' = \frac{1}{2} \cdot \frac{1}{\sqrt{1 + x^2}} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}$$

Now substitute into DE:

$$(1+x^2)\frac{x}{\sqrt{1+x^2}} = xy$$
$$\sqrt{1+x^2} = y$$

The equality holds, $y = \sqrt{1+x^2}$ is a solution.

Verification of implicit solution

Example

Verify that $y^2 - 1 = (x+2)^2$ is a solution of DE $y^2 - 1 - (2y + xy)y' = 0$.

Let us differentiate solution to express y':

$$y^2 - 1 = (x+2)^2$$
$$2yy' = 2x + 4$$
$$y' = \frac{x+2}{y}$$

Now y' can be substituted into DE:

$$y^2 - 1 - (2y + xy)\frac{x+2}{y} = 0$$
$$y^2 - 1 - (2+x)(x+2) = 0$$
$$(y^2 - 1) - (x+2)^2 = 0$$

Since $y^2 - 1$ is equal to $(x+2)^2$, both paranthesis cancel each other and then

$$(x+2)^2 - (x+2)^2 = 0$$
$$0 = 0$$

Therefore $y^2 - 1 = (x+2)^2$ is a solution of DE $y^2 - 1 - (2y + xy)y' = 0$.

Example

Determine, whether

$$e^{2y} + e^{2x} = 1$$

is implicit function which is an implicit solution of differential equation

$$e^{x-y} + e^{y-x}\frac{dy}{dx} = 0$$

First let us differentiate solution to express y':

$$2e^{2y}y' + 2e^{2x} = 0$$
$$y' = \frac{-2e^{2x}}{2e^{2y}} = -\frac{e^{2x}}{e^{2y}}$$

Now let us substitute prepared y' into DE.

$$e^{x-y} + e^{y-x}\frac{-e^{2x}}{e^{2y}} = 0$$
$$\frac{e^x}{e^y} + \frac{e^y}{e^x}\left(-e^x e^x \cdot \frac{1}{e^y} \cdot \frac{1}{e^y}\right) = 0$$
$$\frac{e^x}{e^y} - \left(e^x \cdot \frac{1}{e^y}\right) = 0$$
$$0 = 0$$

Example

Is $x^2 + y^2 + 1 = 0$ an implicit solution of $dy/dx = -x/y$?

If we differentiate proposed solution, we get $2x + 2yy' = 0 \implies y' = -x/y$. That looks well but the trouble here is that $x^2 + y^2 + 1 = 0$ is not a function, so it can not be a solution.

First order DE, higher order DE

The first order DE involves variable differentiated once. If we can observe presence either of y'' or $e^{(4)}$, it is not first order differential equation.

Example

Determine the order of the following DE:

$$dy + (xy - \cos x)dx = 0 \tag{1}$$
$$y'' + xy'' + 2y(y')^3 + xy = 0 \tag{2}$$
$$\left(\frac{d^2y}{dx^2}\right)^3 - (y''')^4 + x = 0 \tag{3}$$
$$e^{y'''} + xy'' + y = 0 \tag{4}$$

Answers: DE (1) is first order DE, DE (2) is second order DE, DE (3) is third order DE, DE (4) is third order DE.

Particular, singular and general solution

As the graphs (direction fields) above suggest, a differential equation does not have one, but many solutions. That is caused by presence of integrating constant(s). Let us for example consider differential equation $y' = x$. Then $y(x) = 1/2x^2$ is a solution of given DE. And $y(x) = 1/2x^2 + c$ is a solution as well.

In the next chapters we will be talking about

1. **General solution:** general solution contains every particular solution, can be also considered as a family of solutions.
2. **Particular solution:** does not contain arbitrary constant c. The arbitrary constant is solved in accordance with initial or boundary values of the problem.
3. **Singular solution:** sometimes, DE has also other general, but unusual solution, which can not be obtained from the family of general solutions.

For example $y = xy' + (y')^2$ is a first order DE with general solution $y = cx + c^2$. However it is not general solution in the real meaning of this term, because it does not include every particular solution. Because the function $y = -x^2/4$ is also a solution, which can not be obtained from the group of solutions no matter what the value c is.

Initial value for DE

Initial values are conditions, which allow us to determine the values of arbitrary constants $c_1, c_2, \ldots, c_n$. Normally, **the number of arbitrary constants equals to the order of DE.** We might say that for the first order DE we need one initial value to determine constant c. For the second order DE we need two initial values to solve both c_1, c_2.

Example

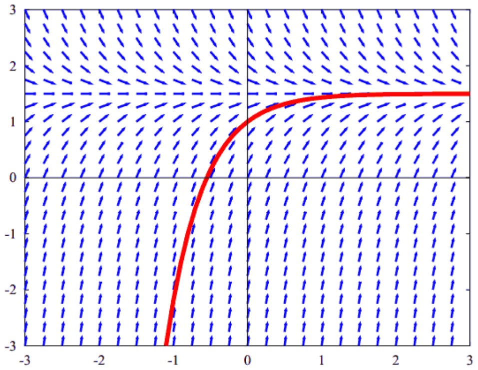
Differential equation $y' + 2y = 3$ has a solution $y = 3/2 + c/e^{2x}$. What is the particular solution if initial value is $y(0) = 1$?

The solution seems to be a general solution and can be drawn by means of direction field. But direction field (and the general solution) expresses a family of solutions. We have to find such solution which fulfills $y(0) = 1$ or—in other words—solution, which passes point $[0, 1]$.

$$\left.\begin{array}{l} x = 0 \\ y = 1 \end{array}\right\} \implies 1 = \frac{3}{2} + \frac{c}{e^{2\cdot 0}} \implies c = -\frac{1}{2}$$

Then the particular solution is

$$y = \frac{3}{2} - \frac{1}{2} \cdot \frac{1}{e^{2x}}$$



Example

Differential equation $yy' = (y+1)^2$ has a family of solutions $1/(y+1) + \log|y+1| = x + c,\ y \neq -1$. Find the particular solution for which $y(2) = 0$.

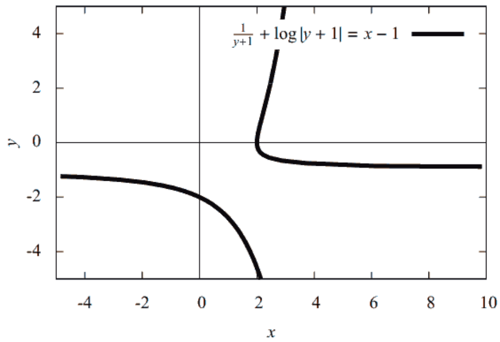
From the family of solutions we have to find the solution which passes point $[2, 0]$:

$$\left.\begin{array}{l} x = 2 \\ y = 0 \end{array}\right\} \implies \frac{1}{0+1} + \log|0+1| = 2 + c \implies c = -1$$

The value of constant c is determined from initial value and particular solution (in implicit form) can be written as:

$$\frac{1}{y+1} + \log|y+1| = x - 1,\ y \neq -1$$

Note: the function defined as $y = -1$, which we have to discard above, is also solution of the given DE and is a *singular solution*.



More problems solved...

Example

Prove that the function $y = ae^x + be^{-x}$ is a solution of differential equation $y'' - y = 0$

Let us get first y'' from the solution:

$$y' = ae^x - be^{-x}$$
$$y'' = ae^x + be^{-x}$$

If we substitute both $y,\ y''$ into DE the equality holds, so $y = ae^x + be^{-x}$ is a solution of given DE.

Example

Show that $y = 2/(x+2),\ x \neq -2$ is a solution of DE $xy' + y = y^2$.

$$y = \frac{2}{x+2}$$
$$y' = 2\left((x+2)^{-1}\right)' = -2(x+2)^{-2}$$

Let us substitute $y,\ y'$ into DE:

$$x(-2(x+2)^{-2}) + \frac{2}{x+2} = \frac{4}{(x+2)^2} \qquad / \cdot (x+2)^2$$
$$-2x + 2x + 4 = 4$$
$$4 = 4$$

The equality holds, so $y = 2/(x+2),\ x \neq -2$ is a solution of the given DE.

Example

Test whether family of solution $y = c_1 + c_2 e^{-x} + x^3/3$ satisfies differential equation $y'' + y' - x^2 - 2x = 0$.

We need $y',\ y''$ from the solution and test, whether, when substituted into DE, equality holds.

$$y = c_1 + c_2 e^{-x} + x^3/3$$
$$y' = -c_2 e^{-x} + x^2$$
$$y'' = c_2 e^{-x} + 2x$$

Let us substitute into DE:

$$(c_2 e^{-x} + 2x) + (-c_2 e^{-x} + x^2) - x^2 - 2x = 0$$
$$0 = 0$$

Example

Find DE whose solution is a family $y = cx + c^3$

There is an arbitrary constant c, which is involved in solution. Since there is only one constant, the DE is of the first order. Let us differentiate solution, express c and substitute into solution. Then we have differential equation.

$$y = cx + c^3$$
$$y' = c$$
$$c = y'$$

Then

$$\underline{\underline{y = y'x + (y')^3}}$$

Example

Find DE whose solution is a family $x^2 - cy + c^2 = 0$

$$
\begin{aligned}
x^2 - cy + c^2 &= 0 \\
2x - cy' &= 0 \implies c = \frac{2x}{y'} \\
x^2 - \frac{2x}{y'}\,y + \frac{4x^2}{(y')^2} &= 0 \\
x^2(y')^2 - 2xyy' + 4x^2 &= 0
\end{aligned}
$$

Example

Prove that $y = e^{-x/2+c}$ is a solution of DE $2y' + y = 0$. Find particular solution if

- $y(0) = 2$,
- $y(0) = -2$.

Make a direction field for DE and draw the particular solutions.

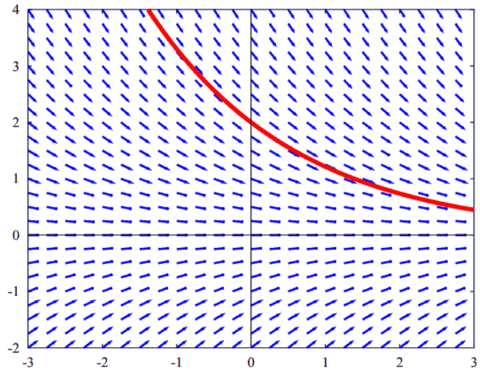
Firstly, let us differentiate supposed solution so it can be substituted into DE.

$$
\begin{aligned}
y = e^{-\frac{x}{2}+c} &\implies y' = -\frac{1}{2}\,e^{-\frac{x}{2}+c} \\
2y' + y &= 0 \\
2(-\frac{1}{2}\,e^{-\frac{x}{2}+c}) + (e^{-\frac{x}{2}+c}) &= 0 \\
0 &= 0
\end{aligned}
$$

The supposed solution is really the solution. Now let us solve arbitrary constant c for given initial values in order to express particular solutions.

$$
\begin{aligned}
y(x=0) = 2: &\qquad 2 = e^{-0+c} \implies c = \log 2 \\
y(x=0) = -2: &\qquad -2 = e^{-0+c}
\end{aligned}
$$

The particular solution for initial value $y(0) = 2$ is $y = e^{-x/2+0.693}$ and is shown in graph. There is no particular solution passing the point $[0, -2]$.



First order differential equations

DE solved by separating variables

We recognize many types of differential equation. Such recognizing is the key for solving, because then we can apply the proper method, which is able to bring the solution of DE.

We know already how to solve simple DE in the form

$$\frac{dy}{dx} = g(x).$$

Just integrating both sides gives us the solution:

$$\begin{aligned} dy &= g(x)\,dx, \qquad (5) \\ y &= \int g(x)\,dx, \\ y &= G(x) + c, \end{aligned}$$

where $G(x)$ is an antiderivative of $g(x)$. Now what if in (5) is on the left side not only dy, but a function depending on y? The solution would be that simple as well. Just to integrate both left and right side.

That means **if we can separate members depending on x together with dx and at the same time do the same for y, then it is easy to solve DE by integrating both sides.** So

$$\frac{dy}{dx} = g(x)\,h(y) \qquad (6)$$

would be the general form of DE, which can be solved by separating variables. For example

$y' = xy$	$y' = xy + \sin(x+y)$
or	or
$y' = x/y$	$y' = x/y + \log(x+y)$
are separable	are **not** separable

Solution without solution (curves, direction field)

In many practical problems a rough geometrical approximation to a solution might be all that is needed. Sometimes we may even be not able to solve the DE symbolically.

We can construct a **direction field:** a rectangular area with grid points and compute/draw tangent dy/dx **for each grid point.** See the graphs within this chapter, e.g. the last example for illustration.

Producing direction field by hand might be time consuming. It is handy if one can find so called **isoclines,** i.e. lines which have the same value of the slope.

Example

Solve DE $x\,dy - y\,dx = 0$. Find particular solution for $y(2) = 4$.

There is not much effort needed to reshuffle DE into

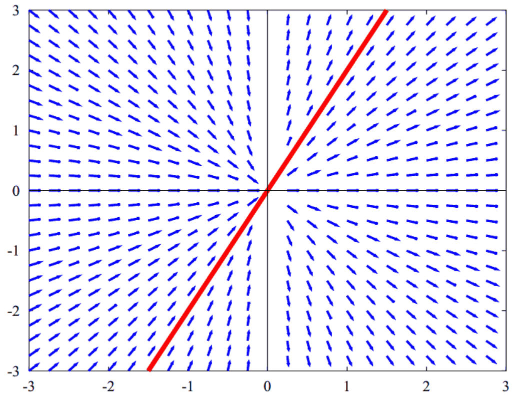
$$\frac{1}{y}\,dy - \frac{1}{x}\,dx = 0$$

The variables x are separated from y. It is simple task to integrate both sides now.

$$\begin{aligned} \log|y| &= \log|x| + c \\ y &= e^{\log|x|+c} \\ y &= e^{\log|x|} \cdot e^{c} \\ y &= xC \end{aligned}$$

To find the particular solution we have to substitute initial value $y(x=2) = 4$ into solution. Then value $C = 2$ is found.

$$\underline{\underline{y = 2x}}$$



Direction field and the particular solution for given DE. Direction fields are small tangents (dy/dx) evaluated and drawn for grids within the network of grids. These small blue arrows provide idea about all solutions. Also one (particular) solution $y = 2x$ is drawn. Note that the solution follows direction field.

Example

Solve DE $yx^2dy - y^3dx = 2x^2dy$.

The DE is somehow more complicated than the previous one. We have to put some effort into separating variables. If we succeed to separate variables, then after integrating we have the solution.

Let us divide by x^2y^3 and see what happens

$$\frac{1}{y^2}\,dy - \frac{1}{x^2}\,dx = 2\,\frac{1}{y^3}\,dy$$

We separated the variables quite easily. Now all members can be integrated.

$$\begin{aligned} -x^{-2}dx &= (2y^{-3} - y^{-2})\,dy \\ x^{-1} &= -y^{-2} + y^{-1} + C \\ \frac{1}{x} &= -\frac{1}{y^2} + \frac{1}{y} + C \qquad x \neq 0,\ y \neq 0 \\ y^2 &= -x + xy + xy^2C \end{aligned}$$

We can use quadratic formula or leave the solution in implicit form.

$$\underline{\underline{y^2(1 - xC) = x(y - 1)}}$$

Note: we have lost a solution. The above family of solutions is not the general solution.

During the operations we ruled out $y = 0$. If we substitute $y = 0$ (and $dy = 0$, since y is constant) into given DE, then it is quite clear that the equality holds for any x. Thus $y = 0$ is also a solution.

$$\underline{\underline{y = 0}} \qquad \text{(singular solution)}$$

Example

Solve DE $y' = e^{\sqrt{x}}/y$ for initial value $y(1) = 4$.

$$y\,dy = e^{\sqrt{x}}dx$$

When the variables are separated, the left side is easy to integrate, but the right side is more complicated.

We have to solve $\int e^{\sqrt{x}}dx$. It contains internal function so we have to use the chain rule backwards. First let us differentiate $e^{\sqrt{x}}$ and then integrate the product by integration by parts (the product rule backwards):

$$(e^{\sqrt{x}})' = \frac{1}{2}\, e^{\sqrt{x}} x^{-1/2}$$

Now to integrate $1/2\, e^{\sqrt{x}} x^{-1/2}$ we use integration by parts. From $(uv)' = uv' + u'v \implies uv' = (uv)' - u'v$ we have $\int uv' = uv - \int u'v$. One member is easy to differentiate ($e^{\sqrt{x}}$) and the second one is easy to integrate ($1/2\, x^{-1/2}$).

$$\begin{aligned} e^{\sqrt{x}} &= \int \frac{1}{2}\, e^{\sqrt{x}} x^{-1/2}\, dx = \\ &= \int uv'\, dx = \left| \begin{matrix} u = e^{\sqrt{x}} & \implies & u' = \frac{1}{2}\, e^{\sqrt{x}} x^{-1/2} \\ v' = \frac{1}{2}\, x^{-1/2} & \implies & v = x^{1/2} \end{matrix} \right| = \\ &= e^{\sqrt{x}} x^{1/2} - \frac{1}{2} \int e^{\sqrt{x}} dx \\ \frac{1}{2} \int e^{\sqrt{x}} dx &= e^{\sqrt{x}} x^{1/2} - e^{\sqrt{x}} \\ \int e^{\sqrt{x}} dx &= 2e^{\sqrt{x}} x^{1/2} - 2e^{\sqrt{x}} \end{aligned}$$

Now we have figured out how to integrate both left and right side, so we can integrate, therefore complete the solution of DE:

$$\begin{aligned} \frac{1}{2}\, y^2 &= 2e^{\sqrt{x}} x^{1/2} - 2e^{\sqrt{x}} + c. \\ y &= 2\sqrt{e^{\sqrt{x}} \sqrt{x} - e^{\sqrt{x}} + C}. \end{aligned}$$

If the initial value $y(x=1) = 4$ is substituted into the general solution, the value of constant is determined as $C = 4$. The particular solution is then

$$y = 2\sqrt{e^{\sqrt{x}} \sqrt{x} - e^{\sqrt{x}} + 4}\,.$$

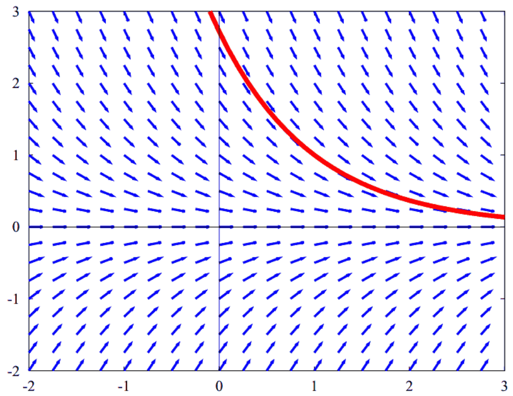
Example

Solve DE $y' + y = 0$ for initial value $y(1) = 1$.

$$\begin{aligned} \frac{1}{y}\, dy + dx &= 0 \\ \log|y| + x &= c \\ \log|y| &= c - x \\ y &= e^{c-x} \end{aligned}$$

Substituting initial value $y(x=1) = 1$ into solution gives $c = 1$

$$y = e^{1-x}$$



Example

Solve DE $\sin x\cos 2y\,dx+\cos x\sin 2y\,dy=0$ for initial value $y(0)=\pi/2$.

$$\begin{aligned}\frac{\sin x}{\cos x}\,dx+\frac{\sin 2y}{\cos 2y}\,dy&=0\\ \tan x\,dx+\tan 2y\,dy&=0\\ -\log|\cos x|-\frac{1}{2}\log|\cos 2y|&=C\\ \log|\cos x|+\frac{1}{2}\log|\cos 2y|&=D\\ \log\sqrt{|\cos 2y|}&=D-\log|\cos x|\\ \sqrt{|\cos 2y|}&=e^{D-\log|\cos x|}\\ \sqrt{|\cos 2y|}&=\frac{e^D}{\cos x}\\ \cos 2y\cdot\cos^2 x&=E\end{aligned}$$

If we use initial value within the solution we get $E=-1$, so the particular solution is

$$\underline{\underline{\cos 2y\cdot\cos^2 x=-1}}$$

Example

Solve DE $dr/d\theta=-\sin\theta$.

$$dr=-\sin\theta\,d\theta$$

After integrating both sides:

$$\underline{\underline{r=\cos\theta+C}}$$

Example

Solve DE $dr/d\theta\cot\theta-r=2$. (Note that $\cot\theta=\cos\theta/\sin\theta$)

$$\begin{aligned}\frac{1}{2+r}\,dr&=\tan\theta\,d\theta\\ \log|2+r|&=\log|\sec\theta|+C\\ e^{\log|2+r|}&=e^{\log|\sec\theta|}+C\end{aligned}$$

Note: function $\sec\theta=1/\cos\theta=\text{hypotenuse}/\text{adjacent}$.

Assuming $\theta\neq\ldots,\ \pi/2,\ 3\pi/2,\ \ldots$ and $r\neq-2$, the general solution is

$$\underline{\underline{2+r=D\sec\theta}}\tag{7}$$

A numerical check of solution for correctness

Now we have reached a solution but when solving DE, esp. difficult ones, it is quite possible we could make a mistake. For that reason, we might be interested **whether there is a method to check the solution for the correctness.**

Well, there is a **simple numerical way of checking by using a calculator.** Let us choose the constant D of any value, e. g. $D=2$ and take any value for θ, e.g. $\theta=1$ and evaluate r from the solution. Then let us choose any *small* $d\theta$, e.g. $d\theta=0.001$.

So, from solution, we have

- $D = 2,\ \theta = 1 \implies r = 1.7016314$
- $D = 2,\ \theta = (1 + d\theta) \implies r = 1.7074072 \implies dr = 0.0057758$

We got two points (from the solution), which are close to each other. Not only we can draw the graph of solution, but **we can numerically evaluate approximate tangent** at a point and note that the tangent is used also within the context of DE **(it is y' within DE indeed)**. If there is no mistake within the solution, we can expect that after substituting prepared numerical values into DE, left and right side have to hold equality (roughly):

$$\frac{dr}{d\theta}\cot\theta - r = 2$$
$$\frac{0.0057758}{0.001}\cot 1 - 1.70741 = 2$$
$$2.0012 \doteq 2$$

Despite all the simplifications we have used, the last line shows that the tangent we got from our solution fits into the DE and that suggests that the solution (7) is correct.

Example

Solve DE with initial values. Draw direction field and solution on $x = \{-3, -2, -1, 0, 1, 2\} \times y = \{-2, -1, 0, 1, 2\}$

$$y' = \frac{2x+1}{2y}, \qquad y(-2) = -1, \qquad y \neq 0$$

After some shuffling and integrating both sides:

$$y^2 = x^2 + x + c$$

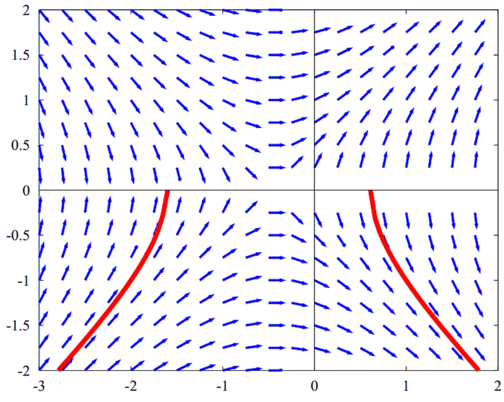
For given initial value then $c = -1$

Now we have implicit solution which can be splitted into two solutions

$$y = +\sqrt{x^2 + x - 1} \qquad \text{and}$$
$$y = -\sqrt{x^2 + x - 1}$$

The initial value fits only to the second one. For drawing the direction field, let us form a table, things go fast and easy then.

	x	-3	-2	-1	0	1	2	3
y								
-2		5/4	3/4	1/4	-1/4	-3/4	-5/4	-7/4
-1		5/2	3/2	1/2	-1/2	-3/2	-5/2	-7/2
0		—	—	—	—	—	—	—
1		-5/2	-3/2	-1/2	1/2	3/2	5/2	7/2
2		-5/4	-3/4	-1/4	1/4	3/4	5/4	7/4



Note: more grid points are shown within the chart than are computed within the table.

First order differential equations

Linear DE

The linear DE of first order can be described as

$$a_1(x)\, y' + a_0(x)\, y = g(x).$$

That means both $y,\ y'$ are involved with coefficients, which are either constant or are a function of x.

Note: DE e.g.

$$y' + \sin y = g(x) \quad \text{or}$$
$$(y')^2 + y = g(x)$$

are **not** linear DE.

We will rather work with **linear DE in the standard form**

$$\frac{dy}{dx} + P(x) \cdot y = f(x). \tag{8}$$

In some cases the DE above, which is linear, can be solved also by separating variables. General method to solve linear DE involves a special function $\mu(x)$. As for now, do not think where this function comes from. If we use such **special function $\mu(x)$ to multiply DE,** it will help us to bring the solution. So let us multiply (8) by $\mu(x)$ to derive the principle:

$$\mu(x)\, \frac{dy}{dx} + \mu(x)P(x) \cdot y = \mu(x)f(x) \tag{9}$$

Now it is important to pay attention to the left side of (9). Because it will be shown that **after multiplying both sides of DE by $\mu(x)$, on the left side of DE is being left $\mu(x)y$ differentiated with respect to x.** Let us differentiate $\mu(x)y$:

$$\frac{d}{dx}\, \mu(x)\, y = \mu(x)\, \frac{dy}{dx} + \frac{d\mu}{dx}y \tag{10}$$

By (10) it was shown that $\frac{d}{dx}\, \mu(x)y$ is on the left side of (9) supposing

$$\frac{d\mu}{dx} = \mu(x)P(x) \tag{11}$$

The DE (11) above can be solved by separating variables and will answer the question where does the function $\mu(x)$ come from.

$$\frac{1}{\mu(x)}\, d\mu = P(x)\, dx$$
$$\log|\mu(x)| = \int P(x)\, dx + c_1$$
$$\mu(x) = c_2 e^{\int P(x)\, dx}$$

It can be shown that the value of the constant has no effect on solution of linear DE. So, we may put $c_2 = 1$:

$$\mu(x) = e^{\int P(x)\, dx}. \tag{12}$$

Since we have identified $\mu(x)$ as $e^{\int P(x)\, dx}$, let us use $\mu(x)$ again to multiply the DE (8). Because it will be shown, that after multiplying, the DE is readily solved. Let us bring back DE in standard form and multiply by integrating factor:

$$\frac{dy}{dx} + P(x) \cdot y = f(x) \qquad / \cdot e^{\int P(x)\,dx}$$

$$\left[e^{\int P(x)\,dx}\right] \frac{dy}{dx} + \left[e^{\int P(x)\,dx}\right] P(x) \cdot y = \left[e^{\int P(x)\,dx}\right] f(x)$$

$$\frac{d}{dx}\left(e^{\int P(x)\,dx} y\right) = e^{\int P(x)\,dx} f(x)$$

The most important observation is that **we have $d/dx(\mu(x)y)$ (as a product of the product rule) on the left side.** Now we have to integrate both sides of DE. Especially the left side is simple and the remaining tasks are not so difficult, which will be seen on examples.

The steps to solve linear DE can be summarized as

1. Bring DE into standard form $\frac{dy}{dx} + P(x) \cdot y = f(x)$.
2. Identify $P(x)$ and form the integrating factor $\mu(x)$ from (12).
3. Multiply both sides of DE by integrating factor $\mu(x)$.
4. Integrating both sides of DE is simple then considering the fact on the left side is $d/dx(\mu(x)y)$ to be found.

Example

Solve DE $dy/dx + 3x^2 y = 6x^2$.

Given DE is linear, already in standard form, so we can locate $P(x) = 3x^2$ to evaluate integrating factor $\mu(x)$:

$$\mu(x) = e^{\int 3x^2\,dx} = e^{x^3}.$$

The integrating factor will be used to multiply both sides of DE in the standard form. Then:

$$e^{x^3} \frac{dy}{dx} + e^{x^3}\, 3x^2 y = e^{x^3}\, 6x^2$$

We know in advance that what will be found on the left side is a product of product rule of $d/dx(\mu(x)\,y) = d/dx(e^{x^3} y)$:

$$\frac{d}{dx}\, e^{x^3} y = e^{x^3}\, 6x^2$$

Now all we need is to integrate both sides and especially the left side is trivial.

$$\underline{\underline{e^{x^3} y = 2e^{x^3} + C}}$$

Example

Solve DE $x^2 y' + xy = 1,\ x > 0,\ y(1) = 2.$

The first step is to bring the DE into standard form, then locate $P(x)$ in order to evaluate integrating factor $\mu(x)$:

$$y' = \frac{1}{x} y = \frac{1}{x^2} \implies P(x) = \frac{1}{x} \qquad x \neq 0$$

$$\mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$$

Integrating factor has been found. Now let us use it to multiply DE:

$$y' x + y = \frac{1}{x}$$

On the left side is product of the product rule applied on $d/dx(\mu(x)y) = d/dx(xy) = (xy)'$ as anticipated. So it is easy to integrate both sides then.

$$
\begin{aligned}
(xy)' &= \frac{1}{x} \\
xy &= \log|x| + C \\
y &= \frac{\log|x| + C}{x}
\end{aligned}
$$

That is general solution, constant C has to be found for supplied IV $y(x=1)=2$:

$$2 = \frac{\log|1| + C}{1} \implies C = 2$$

The particular solution is then

$$\underline{\underline{y = \frac{\log x + 2}{x}}}.$$

Example

Solve DE $y' + 2xy = 1$.

The given DE is already in the standard form, so we can find the integrating factor $\mu(x)$ for $P(x) = 2x$:

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}.$$

Let us use $\mu(x)$ to multiply each part of DE:

$$y'e^{x^2} + 2xye^{x^2} = e^{x^2}$$

Again, what is on the left side is $(\mu(x)y)'$, thus trivial to integrate.

$$
\begin{aligned}
(e^{x^2}y)' &= e^{x^2} \\
e^{x^2}y &= \int e^{x^2}\,dx + C
\end{aligned}
$$

However this time on the right side is a function which antiderative can not be expressed in terms of elementary functions. So it is OK to leave the member as $\int e^{x^2}\,dx$.

$$\underline{\underline{y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}}}$$

Example

Solve DE $xy' + y = x^3$.

If we compare this DE with linear DE (8) in standard form, we observe that given DE fits into linear DE if is divided by x:

$$y' + \frac{1}{x}y = x^2.$$

Now it is clear that $P(x) = 1/x$ and we can assemble integrating factor $\mu(x)$ which will be used to multiply both sides of the DE.

$$
\begin{aligned}
\mu(x) &= e^{\int P(x)\,dx} = e^{\int 1/x\,dx} = e^{\log x} \implies \\
\implies \mu(x) &= x
\end{aligned}
$$

As reminded, now the integrating factor $\mu(x) = x$ is good to multiply DE and then $(\mu(x)y)'$ appears on the left side.

$$\begin{aligned} xy' + y &= x^3 \\ (\mu(x)\, y)' = (xy)' &= x^3 \\ xy &= \int x^3 \, dx \end{aligned}$$

Not much can be done when the right side is finally integrated. The solution will be left in implicit form.

$$\underline{\underline{xy = \frac{1}{4}\, x^4 + C}}$$

Example

Solve DE $y' + ay = b$.

DE is in the standard form, now $P(x) = a$, integrating factor

$$\begin{aligned} \mu(x) &= e^{\int P(x)\, dx} = e^{\int a\, dx} = e^{ax} \\ a^{ax}\, \frac{dy}{dx} + e^{ax} ay &= a^{ax} b \\ \frac{d}{dx}\, e^{ax} y &= e^{ax} b \\ e^{ax} y &= \int e^{ax} b \, dx \\ e^{ax} y &= b\, \frac{1}{a}\, e^{ax} + C \end{aligned}$$

And after we divide by e^{ax}:

$$\underline{\underline{y = \frac{b}{a} + ce^{-ax}.}}$$

Example

Solve DE $dy/dx = 2x - 3y,\ y(0) = 1/3$.

First we have to shuffle the members to bring DE into standard form (9) of linear DE.

$$\begin{aligned} \frac{dy}{dx} + 3y =& 2x \\ \mu(x) = e^{\int P(x)\, dx} &= e^{\int 3\, dx} = e^{3x} \\ e^{3x}\, \frac{dy}{dx} + e^{3x} \cdot 3y &= e^{3x} \cdot 2x \end{aligned}$$

Again, after multiplying by $\mu(x)$, on the left side is $(\mu(x) y)'$ to be found.

$$\begin{aligned} \frac{d}{dx}\, e^{3x} y &= e^{3x} \cdot 2x \\ e^{3x} y &= 2 \int e^{3x} x \, dx \end{aligned}$$

To integrate $e^{3x} x$ we have to use integration by parts. Integration by parts is product rule used backwards.

$$(uv)' = u'v + uv' \implies uv' = (uv)' - u'v \implies \int uv' = uv - \int u'v$$

For integration by parts there have to be one member which is easy to integrate and the other one which is easy to differentiate. In this case we differentiate x and integrate e^{3x}.

$$\int e^{3x} x \, dx = \begin{vmatrix} u = x & u' = 1 \\ v' = e^{3x} & v = \frac{1}{3} e^{3x} \end{vmatrix} = x \, \frac{1}{3} \, e^{3x} - \int \frac{1}{3} \, e^{3x} \, dx = \frac{1}{3} \, e^{3x} x - \frac{1}{9} \, e^{3x}$$

$$e^{3x} y = 2(\frac{1}{3} \, e^{3x} x - \frac{1}{9} \, e^{3x}) + C$$

$$y = \frac{2}{3} \, (x - \frac{1}{3}) + Ce^{-3x}$$

Considering given IV $y(x = 0) = 1/3$:

$$\frac{1}{3} = \frac{2}{3} \, (0 - \frac{1}{3}) + Ce^{-3 \cdot 0} \implies C = \frac{5}{9}$$

Applying evaluated constant into the general solution we get particular solution for IV:

$$y = \frac{2}{3} \, (x - \frac{1}{3}) + \frac{5}{9} \, e^{-3x} .$$

Example

Solve DE $xy' + y = e^x, \; y(1) = 2$.

DE is not in the standard form yet.

$$y' + \frac{1}{x} \, y = \frac{1}{x} \, e^x \implies P(x) = 1/x$$

$$\mu(x) = e^{\int P(x) \, dx} = e^{\int 1/x \, dx} = e^{\log x} = x$$

The integrating factor $\mu(x)$ was needed to multiply the DE, because that is the way how to solve it:

$$xy' + x \, \frac{1}{x} \, y = x \, \frac{1}{x} \, e^x$$

$$xy' + y = e^x$$

$$\frac{d}{dx} \, xy = e^x$$

As expected, on the left side is $d/dx \; \mu(x)y = d/dx \; xy$, which when integrated become xy:

$$(xy)' = e^x$$

$$xy = \int e^x \, dx$$

$$xy = e^x + C$$

$$y = \frac{1}{x} \, (e^x + C)$$

Applying IV $y(x = 1) = 2$ brings value of the constant $C = (2 - e)$ and the particular solution is

$$y = \frac{1}{x} \, (e^x + 2 - e) .$$

First order differential equations

Exact (total) DE

We are going to deal with such types of differential equations which can be seen as a product of exact (total) differential. For example let us have a function

$$f(x,y) = 3x^2 + 4xy + y, \qquad \text{then}$$
$$df = \frac{\partial}{\partial x} f(x,y)\,dx + \frac{\partial}{\partial y} f(x,y)\,dy =$$
$$= (6x+4y)\,dx + (4x+1)\,dy$$

is called total differential of $f(x,y)$ and such observation helps us to solve differential equation

$$(6x+4y)\,dx + (4x+1)\,dy = 0. \tag{13}$$

The first thought to solve (13) is to go back against differentiation, i.e. to integrate both members. But the case is not the simple, since some members repeat:

$$\int (6x+4y)\,dx = 3x^2 + 4xy \qquad \text{and}$$
$$\int (4x+1)\,dy = 4xy + y.$$

So, that was an ilustrative example to outline the kind of DE we are dealing with within this chapter. Now let us talk more in general. We are going to solve DE in **standard form** of

$$M(x,y)\,dx + N(x,y)\,dy = 0, \qquad \text{where}$$
$$M(x,y) = \frac{\partial}{\partial x} f(x,y) \quad \text{and} \quad N(x,y) = \frac{\partial}{\partial y} f(x,y).$$

If the members $M(x,y)\,dx + N(x,y)\,dy$ are results of total differential, we can solve DE by finding $f(x,y)$ from which members $M(x,y)$, $N(x,y)$ were derived.

Before we can go after the solution, **we have to test whether $M(x,y)$, $N(x,y)$ are really products of total differential** of $f(x,y)$. If the equality

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y)$$

holds then the differential equation is exact and can be solved as an exact differential equation.

The next step is to find the solution itself. If we integrate $M(x,y)$ and $N(x,y)$ we can reconstruct $f(x,y)$. But as was shown, some members can come twice: once from $\int M(x,y)\,dx$, second from $\int N(x,y)\,dy$. To solve these troubles, let us integrate $M(x,y)$ first and let it equal to $f(x,y)$, which is what we are looking for:

$$f(x,y) = \int M(x,y)\,dx + g(y) \tag{14}$$

We are near the result $f(x,y)$, but we have to evaluate also $g(y)$ which is a constant coming as a product of integration. It comes from definition that if we differentiate $f(x,y)$ with respect to y, we obtain $N(x,y)$:

$$N(x,y) = \frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y} \int M(x,y)\,dx + g'(y)$$

Since $N(x,y)$ is given by differential equation and $\frac{\partial}{\partial y} \int M(x,y)\,dx$ is quite easy to evaluate, we can express wanted $g(y)$ as

$$g'(y) = N(x,y) - \frac{\partial}{\partial y} \int M(x,y)\,dx \tag{15}$$

The task remains to integrate $g'(y)$ obtained from (15) and substitute $g(y)$ to (14). Then the solution sums $\int M(x,y)\,dx$ (in (14)) and $\int N(x,y)\,dy$ (from (15)) as was anticipated from the beginning. And since some members would appear twice, we have to evaluate and subtract them: these come from $\int M(x,y)\,dx$ differentiated with respect to y in (15).

Note: above is the formal solution. **The intuitive solution is just to integrate both $M(x,y)$ and $N(x,y)$ and involve repeating members only once.**

Example

Solve $2xy - 9x^2 + (2yx^2 + 1)dy/dx = 0$.

First let us bring equation into standard form in order to locate $M(x,y)$, $N(x,y)$.

$$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$$

Test, whether it is exact DE:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy - 9x^2) = 2x, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2y + x^2 + 1) = 2x.$$

The test has confirmed we can solve as an exact DE.

$$\int M(x,y)\,dx = \int (2xy - 9x^2)\,dx = x^2y - 3x^3 + g(y)$$
$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx = (2y + x^2 + 1) - \frac{\partial}{\partial y}(x^2y - 3x^3) = 2y + 1$$
$$g(y) = y^2 + y$$

Then the solution is

$$\underline{\underline{f(x,y) = (x^2y - 3x^3) + (y^2 + y) = C}}.$$

Alternative solution by intuition:

$$\int M(x,y)\,dx = \int (2xy - 9x^2)\,dx = x^2y - 3x^3 + C_1$$
$$\int N(x,y)\,dy = \int (2y + x^2 + 1)\,dy = y^2 + x^2y + y + C_2$$

The product x^2y comes twice from both $\int M\,dx$ and $\int N\,dy$. We locate such members and include them to solution only once:

$$\underline{\underline{f(x,y) = x^2y - 3x^3 + y^2 + y = C}}.$$

Example

Solve $2xy^2 + 4 = 2(3 - x^2y)y'$, $y(-1) = 8$.

First let us bring equation into standard form in order to locate $M(x,y)$, $N(x,y)$.

$$2xy^2 + 4 = 2(3 - x^2y)\frac{dy}{dx}$$
$$(2xy^2 + 4)\,dx + 2(x^2y - 3)\,dy = 0$$

First test whether it is exact DE:

$$\frac{\partial M}{\partial y} = 4xy, \qquad \frac{\partial N}{\partial x} = 4xy.$$

Now we can solve as exact DE.

$$\int M(x,y)\,dx = x^2y^2 + 4x + C_1$$
$$\int N(x,y)\,dy = -6y + x^2y^2 + C_2$$

DE has the solution

$$x^2y^2 + 4x - 6y = C.$$

Initial values $y(-1) = 8$:

$$(-1)^2 \cdot 8^2 + 4(-1) - 6 \cdot 8 = C \implies C = 12$$

Particular solution is then

$$\underline{\underline{x^2y^2 + 4x - 6y = 12}}.$$

Example

Solve $\frac{2ty}{t^2+1} - 2t - (2 - \log(t^2+1))\frac{dy}{dt} = 0,\ y(5) = 0.$

First let us bring equation into standard form in order to locate $M(t,y)$, $N(t,y)$.

$$\frac{2ty}{t^2+1}\,dt - 2t\,dt - (2 - \log(t^2+1))\,dy = 0$$
$$dt\,\Big(\frac{2ty}{t^2+1} - 2t\Big) + dy\,(\log(t^2+1) - 2) = 0$$

The test for exactness says that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} = \frac{2t}{t^2+1}\,.$$

The DE is exact so we can continue with integrating.

$$\int M(t,y)\,dt = \int \Big(\frac{2ty}{t^2+1} - 2t\Big)\,dt = 2\int \Big(y\,\frac{t}{t^2+1} - t\Big)\,dt =$$
$$= y\log(t^2+1) - t^2 + h(y)$$
$$\int N(t,y)\,dy = \int (\log(t^2+1) - 2)\,dy = y\log(t^2+1) - 2y + g(t) \qquad (16)$$

The implicit solution is

$$f(t,y) = y\log(t^2+1) - t^2 - 2y = c.$$

For initial value of $t = 5,\ y = 0 \implies c = 25.$

$$\underline{\underline{y\log(t^2+1) - t^2 - 2y = -25}}$$

is particular solution for $y(5) = 0$.

Example

Solve $(x - 2xy + e^y)\,dx + (y - x^2 + xe^y)\,dy = 0.$

The test for exactness says that

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = -2x + e^y.$$

We can solve as an exact DE:

$$\int M(x,y)\,dx = \frac{x^2}{2} - x^2y + xe^y + g(y)$$
$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx = (y - x^2 + xe^y) - (-x^2 + xe^y) = y$$
$$g(y) = \frac{1}{2}\,y^2$$

The general solution is

$$\underline{\underline{f(x,y) = \frac{x^2}{2} - x^2y + xe^y + \frac{y^2}{2} + C.}}$$

Example

Solve $\cos y\,dx - (x\sin y - y^2)\,dy = 0$.

The test for exactness says that

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = -\sin y.$$

We can solve as an exact DE:

$$\int M(x,y)\,dx = \int \cos y\,dx = \cos y \cdot x + g(y)$$
$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx = -x\sin y + y^2 - (-\sin y \cdot x) = y^2$$
$$g(y) = \frac{1}{3}\,y^3$$

The general solution is

$$\underline{\underline{\cos y \cdot x + \frac{1}{3}\,y^3 = c.}}$$

Example

Solve $(4x^3 - \sin x + y^3)\,dx - (y^2 + 1 - 3xy^2)\,dy = 0$.

The test for exactness says that

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = 3y^2.$$

We can solve as an exact DE:

$$\int M(x,y)\,dx = \int (4x^3 - \sin x + y^3)\,dx = x^4 + \cos x + xy^3$$
$$\int N(x,y)\,dy = -\int (y^2 + 1 - 3xy^2)\,dy = -\frac{1}{3}\,y^3 - y + xy^3$$

The general solution is

$$\underline{\underline{0 = x^4 + \cos x - \frac{1}{3}\,y^3 - y + xy^3 + c.}}$$

First order differential equations

Solution by substitution

We often convert complicated differential equations into simple differential equations by substitution. We will deal with two types of substitutions:

- **Homogeneous differential equation,** where substitution either $u = y/x$ or $u = x/y$ brings DE into simplier one. E.g.

$$\frac{dy}{dx} - \frac{y}{x} - \sin\frac{y}{x} = 0$$
$$(1 + \frac{x^2}{y^2})\, dx + (1 - \frac{y}{x})\, dy = 0$$

- **Bernoulli's differential equation.** This kind can be converted into linear DE by eliminating y^n on the right side.

$$\frac{dy}{dx} + P(x)y = f(x) \cdot y^n$$

First order linear differential equations

Solution by substitution—homogeneous DE

Let us have differential equations e.g.

$$\frac{dy}{dx} - \frac{y}{x} - \sin\frac{y}{x} = 0$$
$$(1 + \frac{x^2}{y^2})\,dx + (1 - \frac{y}{x})\,dy = 0$$

Both above DE can be solved by means of substitution $u = x/u$ or $u = y/x$.

Let us talk now about DE in the standard form of

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same order, then the substitution will lead to a differential equation, in which the variables are separable.

So we have to explain the term **homogeneous** as well. If $M(x,y)$ and $N(x,y)$ can be written in form

$$M(x,y) = x^n \cdot g_1(u)$$
$$N(x,y) = x^n \cdot g_2(u) \qquad \text{where } u = \frac{y}{x}$$

then both members are homogeneous of order n. The substitution written as $u = y/x$ is equivalent to $y = u \cdot x$, so we have to use

$$dy = du\,x + u\,dx$$

Note: sometimes it is preferable to use rather $u = x/y$. In such case the test proving that substitution leads to DE with separable variables is

$$M(x,y) = y^n \cdot g_1(u)$$
$$N(x,y) = y^n \cdot g_2(u) \qquad \text{where } u = \frac{x}{y}$$

Example

Solve $2xy\,dx + (x^2 + y^2)\,dy = 0$ by substitution $u = y/x$.

It is a good practice to test first, whether DE is really homogeneous.

$$x^2(2\,\frac{y}{x})\,dx + x^2(1 + \frac{y^2}{x^2})\,dy = 0$$

We can locate above both members $M(x,y)$ and $N(x,y)$ and it is shown that they are homogeneous of the 2nd order.

$$M(x,y) = x^2 \cdot g_1(u)$$
$$N(x,y) = x^2 \cdot g_2(u)$$

$$y = \frac{y}{x} \implies y = ux \implies dy = du\,x + u\,dx$$

Now we can apply the substitution and solve DE.

$$
\begin{aligned}
2\,\frac{y}{x}\;dx + (1+\frac{y^2}{x^2})\;dy &= 0\\
2u\;dx + (1+u^2)(du\;x + u\;dx) &= 0\\
2u\;dx + u\;dx + du\;x + u^2x\;du + u^3\;dx &= 0\\
dx(3u+u^3) + du\;x(1+u^2) &= 0\\
\frac{1}{x}\;dx + \frac{1+u^2}{3u+u^3}\;du &= 0
\end{aligned}
$$

Note that $(\log|3u+u^3|)' = (3+3u^2)/(3u+u^3)$.

$$
\begin{aligned}
\log|x| + \frac{1}{3}\log|u^3+3u| &= c\\
\log|x| &= -\frac{1}{3}\log|u^3+3u| + c\\
x &= \frac{c}{\sqrt[3]{u^3+3u}}\\
x^3 &= \frac{c}{u^3+3u}\\
x^3u^3 + x^3\cdot 3u &= c\\
x^3\,\frac{y^3}{x^3} + x^3\cdot 3\,\frac{y}{x} &= c
\end{aligned}
$$

The last one can be simplified to

$$\underline{\underline{y^3 + 3x^2y = c}}.$$

Example

Solve $(x+y)\;dx - (x-y)\;dy = 0$ by substitution $u = y/x$.

It is a good practice to test first, whether DE is really homogeneous:

$$x(1+\frac{y}{x})\;dx - x(1-\frac{y}{x})\;dy = 0$$

Voluntary test shows that both $M(x,y)$ and $N(x,y)$ are homogeneous of the first order and we can use the substitution to convert DE into DE, which can be solved by separating variables.

Since $u = y/x,\; y = ux \implies dy = du\;x + u\;dx$.

$$
\begin{aligned}
(x+ux)\;dx - (x-ux)(du\;x + u\;dx) &= 0\\
dx\;x(1+u^2) + du\;(x^2)(u-1) &= 0\\
\frac{1}{x}\;dx + \frac{u-1}{1+u^2}\;du &= 0\\
\frac{1}{x}\;dx + (\frac{u}{1+u^2} - \frac{1}{1+u^2})\;du &= 0\\
\log|x| + \frac{1}{2}\log(1+u^2) - \arctan u &= c\\
\frac{1}{2}\log x^2 + \frac{1}{2}\log(1+u^2) - \arctan u &= c\\
\frac{1}{2}\log(x^2 + x^2u^2) &= \arctan u + c
\end{aligned}
$$

And if we go back against the substitution $u = y/x$ we have the final answer

$$\underline{\underline{\frac{1}{2}\log(x^2+y^2) = \arctan\frac{y}{x} + c}}.$$

Example

Solve $xy' - y - x\sin(y/x) = 0$ by substitution $u = y/x$.

First, let us show that the DE is homogeneous:

$$x\,dy + (-y - x\sin x\frac{x}{y}x)\,dx = 0$$
$$x\,dy + x(-\frac{y}{x} - \sin x\frac{x}{y}x)\,dx = 0$$

The DE can be recognized as a homogeneous of the first order. If $u = y/x$ then $y = ux \implies dy = du\,x + u\,dy$.

$$\frac{du\,x + u\,dx}{dx} - u - \sin u = 0$$
$$du\,x + u\,dx - u\,dx - \sin u\,dx = 0$$
$$-\frac{1}{\sin u}\,du + \frac{1}{x}\,dx = 0$$

The integrating of $\int 1/\sin u\,du = -\log|\cos\frac{u}{2}| + \log|\sin\frac{u}{2}|$ is shown here as a result only with no steps.

$$\log|\cos\frac{y}{2x}| - \log|\sin\frac{y}{2x}| + \log|x| + c = 0$$
$$\log|\tan\frac{y}{2x}| - \log|x| - c = 0$$
$$\tan\frac{y}{2x} = xc$$

We can form an explicit solution

$$\underline{\underline{y = 2x\arctan(xc)}}.$$

Example

Solve $(2x^2y + y^3)\,dx + (xy^2 - 2x^3)\,dy = 0$ by substitution $u = x/y$.

Firstly, we should bring the DE into such form, that the homogeneous type of DE can be recognized:

$$y^3(2\frac{x^2}{y^2} + 1)\,dx + y^3(\frac{x}{y} - 2\frac{x^3}{y^3})\,dy = 0$$

Since we can locate y^3 in front of both members $M(x,y)$, $N(x,y)$ which depend on $g_1(u)$ and $g_2(u)$, the given DE is homogeneous of the third order.

$$(2u^2 + 1)\,dx + (u - 2u^3)\,dy = 0$$

The trouble is we have variable u together with $dx,\ dy$. Since $x = uy$, $dx = du\,y + dy\,u$. Then

$$(2u^2 + 1)(du\,y + dy\,u) + (u - 2u^3)\,dy = 0$$

Now it is DE which can be solved by separating variables.

$$
\begin{aligned}
2u^2\,du\,y + du\,y + 2u^3\,dy + u\,dy + u\,dy - 2u^3\,dy &= 0\\
2u^2\,du\,y + y\,du + 2u\,dy &= 0\\
du(2u^2y + y) + 2u\,dy &= 0\\
du\,y(2u^2 + 1) + 2u\,dy &= 0\\
du\,\frac{2u^2+1}{u} + \frac{2}{y}\,dy &= 0\\
du(2u + \frac{1}{u}) + \frac{2}{y}\,dy &= 0\\
u^2 + \log|u| + 2\log|y| &= c\\
u^2 + \log|uy^2| &= c\\
\frac{x^2}{y^2} + \log|\frac{x}{y}\,y^2| &= c
\end{aligned}
$$

The solution has to be expressed in implicit form

$$
\underline{\underline{\frac{x^2}{y^2} + \log|xy| = c}} \qquad x \neq 0,\; y \neq 0
$$

Example

Solve $\frac{y}{x}\cos\frac{y}{x}\;dx - (\frac{x}{y}\sin yx + \cos yx)\;dy = 0$ by substitution $u = y/x$.

If we substitute $u = y/x$ into DE it can be observed that it is homogeneous DE of degree of zero.

$$
\begin{aligned}
u\cos u\cdot dx - (u^{-1}\sin u + \cos u)(du\,x + u\,dx) &= 0\\
u\cos u\cdot dx - du\,xu^{-1}\sin u - u^{-1}\sin u\cdot u\,dx - \cos u\cdot du\,x - \cos u\cdot u\,dx &= 0\\
-\cos u\cdot du\,x - du\,x\,u^{-1}\sin u - u^{-1}\sin u\cdot u\,dx &= 0\\
dx(-u^{-1}\sin u\cdot u) + du(-\cos u\cdot x - xu^{-1}\sin u) &= 0\\
\frac{1}{x}\,dx + du(\frac{-\cos u - u^{-1}\sin u}{-\sin u}) &= 0\\
\frac{1}{x}\,dx + du(\cot u + \frac{1}{u}) &= 0\\
\log|x| + \log|u| + \log|\sin u| &= c\\
\log|xu\sin u| &= c\\
\log|x\,\frac{y}{x}\sin\frac{y}{x}| &= c
\end{aligned}
$$

The last one can be simplified to

$$
\underline{\underline{y\,\sin\frac{y}{x} = C.}}
$$

Example

Solve $x^2 dy/dx - xy = y^2$.

$$
\begin{aligned}
x^2\,dy - xy\,dx &= y^2\,dx\\
x^2\,dy &= (y^2 + xy)\,dx\\
y^2(\frac{x^2}{y^2})\,dy &= y^2(1 + \frac{x}{y})\,dx
\end{aligned}
$$

We have obtained the DE which is homogeneous of degree of 2. The substitution is $u = x/y \implies dx = du\,y + u\,dy$.

$$
\begin{aligned}
u^2 dy &= (1+u)\,dx \\
u^2 dy &= (1+u)(du\,y + u\,dy) \\
u^2 dy &= du\,y + u\,dy + u\,du\,y + u^2\,dy \\
-u\,dy &= du(y+uy) \\
-u\,dy &= du\,y(1+u) \\
-\frac{1}{y}\,dy &= du\,\frac{1+u}{u} \\
-\log|y| &= \log|u| + u + c \\
\log|y| + \log|\frac{x}{y}| &= -\frac{x}{y} + C \\
\log x &= -\frac{x}{y} + C \\
\log x - C &= -\frac{x}{y}
\end{aligned}
$$

Then

$$
\underline{\underline{y = \frac{x}{C - \log x}}}.
$$

Example

Solve $dy/dx = (x^2+y^2)/(2x^2)$.

Note, sometimes $u = y/x$ is convenient, sometimes rather $u = x/y$ is convenient. In this case better to use $u = y/x \implies dy = du\,x + u\,dx$.

$$
\begin{aligned}
dy \cdot 2x^2 &= dx \cdot x^2(\frac{y^2}{x^2} + 1) \\
2x\,du + 2u\,dx &= (u^2+1)\,dx \\
2x\,du + 2u\,dx &= (u^2\,dx) + dx \\
2du\,x &= dx(u^2 - 2u + 1) \\
\frac{1}{u^2-2u+1}\,du &= \frac{1}{2}\cdot\frac{1}{x}\,dx \\
\frac{1}{(u-1)^2}\,du &= \frac{1}{2}\cdot\frac{1}{x}\,dx \\
(u-1)^{-2} du &= \frac{1}{2}\cdot\frac{1}{x}\,dx
\end{aligned}
$$

Let us integrate both sides.

$$
\begin{aligned}
-(u-1)^{-1} &= \frac{1}{2}\log|x| + c \\
-(\frac{y}{x} - 1)^{-1} &= \frac{1}{2}\log|x| + c \\
-\frac{1}{\frac{y}{x} - \frac{x}{x}} &= \frac{1}{2}\log|x| + c \\
-\frac{1}{\frac{y-x}{x}} &= \frac{1}{2}\log|x| + c \\
\frac{x}{x-y} &= \frac{1}{2}\log|x| + c \\
x &= (\frac{1}{2}\log|x| + c)(x-y) \\
\frac{x}{\frac{1}{2}\log|x| + c} &= x - y \\
y &= x - \frac{x}{\frac{1}{2}\log|x| + c}
\end{aligned}
$$

Then y can be expressed as

$$\underline{\underline{y = x(1 - \frac{1}{\frac{1}{2}\log|x| + c})}}.$$

First order differential equations

Solution by substitution—Bernoulli's DE

Let us have DE

$$\frac{dy}{dx} + P(x)y = f(x) \cdot y^n \tag{17}$$

Such DE reminds us linear DE except the member y^n. For $n = 1$ it is solvable by separating variables. For other n it **can be transformed into linear DE by means of substitution**

$$u = y^{1-n}. \tag{18}$$

If we **multiply DE by**

$$(1-n)y^{-n} \tag{19}$$

the problematic member y^n on the right side will be canceled (note/hint: you may realize relationship between above $u = y^{1-n}$ and $(1-n)y^{-n}$; just differentiate u with respect to y.)

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)y^{-n}P(x)y = (1-n)y^{-n}f(x)y^n \tag{20}$$

$$\frac{d}{dx}y^{1-n} + (1-n)y^{-n}P(x)y = (1-n)f(x) \tag{21}$$

$$\frac{du}{dx} + (1-n)y^{-n}P(x)y = (1-n)f(x) \tag{22}$$

From (20) to (21) we go against chain rule (from (21) to (20) there is y differentiated with respect to x).

From (21) to (22) we have to recognize just the substitution (18) $u = y^{1-n}$ itself.

The DE (17) has been transformed into linear DE.

Example

Solve $y' + xy = x \cdot y^{-3}, \quad y \neq 0$.

We have to multiply the DE by y^3 to eliminate y^{-3} on the right side. Let us follow the proposed way of using substitution $u = y^{1-n} = y^{1-(-3)} = y^4$. Now let us multiply both sides of DE by $u' = 4y^3$ (see (19)).

$$4y^3y' + 4y^4x = 4x$$
$$\frac{du}{dx} + 4xu = 4x$$

We transformed given DE into linear DE which is solved by means of integrating factor $\mu(x)$:

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int 4x\,dx} = e^{2x^2}$$
$$e^{2x^2}\frac{du}{dx} + e^{2x^2}4ux = e^{2x^2}4x$$

Since we are solving linear DE, we have an experience that on the left side is $d/dx\ \mu(x)u$ to be found:

$$\frac{d}{dx}(e^{2x^2})u = e^{2x^2}4x$$
$$e^{2x^2}u = \int e^{2x^2}4x\,dx$$
$$e^{2x^2}u = e^{2x^2} + c$$
$$u = 1 + ce^{-2x^2}$$

Now let us go against the substitution to get final solution.

$$\underline{\underline{y^4 = 1 + ce^{-2x^2}}}$$

Example

Solve $xy' + y = y^2 \log x$.

First let us bring DE into the standard form (17):

$$\frac{dy}{dx} = \frac{1}{x}\,y = y^2\,\frac{\log x}{x}$$

We need to eliminate y^2 on the right side to convert DE into linear DE. Let us follow the proposed way of using substitution $u = y^{1-n} = y^{1-2} = y^{-1}$. Now let us multiply both sides of DE by $u' = -y^{-2}$ (see (19)).

$$\begin{aligned} -y^{-2}\,\frac{dy}{dx} - y^{-2}\,\frac{1}{x}\,y &= -\,\frac{\log x}{x} \\ \frac{d}{dx}\,u - y^{-1}\,\frac{1}{x} &= -\,\frac{\log x}{x} \\ \frac{d}{u}\,dx - \frac{1}{x}\,u &= -\,\frac{\log x}{x} \end{aligned}$$

We have linear DE with integrating factor $P(x) = -1/x$:

$$\mu(x) = e^{\int P(x)\,dx} = e^{-\int \frac{1}{x}\,dx} = e^{-\log x} = x^{-1}$$

Now let us use integrating factor to multiply both sides:

$$\begin{aligned} x^{-1}\,\frac{du}{dx} - x^{-1}\,\frac{1}{x}\,u &= -x^{-1}\,\frac{\log x}{x} \\ \frac{d}{dx}\,(x^{-1})u &= -\,\frac{\log x}{x^2} \\ x^{-1}u &= \frac{\log x}{x} + \frac{1}{x} + c \\ u &= \log x + 1 + cx \\ y^{-1} &= \log x + 1 + cx \\ 1 &= y\log x + y + cxy \end{aligned}$$

That might be simplified into

$$\underline{\underline{y = \frac{1}{\log x + 1 + cx}}}$$

Example

Solve $dy/dx + 6y = 30e^{3x}y^{2/3}$.

The DE is already in the standard form (17). In order to eliminate $y^{2/3}$ on the right side, we have to multiply whole DE by $y^{-2/3}$. We will use substitution $u = y^{1-n} = y^{(1-2/3)} = y^{1/3}$. Then we have to multiply DE by $u' = 1/3\,y^{-2/3}$ (check (19)).

$$\begin{aligned} \frac{1}{3}\,y^{-2/3}\,\frac{dy}{dx} + \frac{1}{3}\,y^{-2/3}6y &= 30e^{3x}y^{2/3}\,\frac{1}{3}\,y^{-2/3} \\ \frac{du}{dx} + 2y^{1/3} &= 10e^{3x} \\ \frac{du}{dx} + 2u &= 10e^{3x} \end{aligned}$$

Now it is linear DE with $P(x) = 2 \implies \mu(x) = e^{2x}$. Let us multiply DE by the integrating factor, then we know that on the left side we will find $d/dx\ \mu(x)u$:

$$\begin{aligned}\frac{d}{dx}\, e^{2x} u &= e^{2x} \cdot 10 e^{3x} \\ e^{2x} u &= \frac{10}{5}\, e^{5x} + c \\ u &= 2e^{3x} + \frac{c}{e^{2x}} \\ y^{1/3} &= 2e^{3x} + ce^{-2x}\end{aligned}$$

Then the solution is

$$\underline{\underline{y = (2e^{3x} + ce^{-2x})^3}}$$

Example

Solve $6y' - 2y = xy^4, \quad y(0) = -2.$

It can be observed that the DE is of Benoulli type. However it is needed to bring the DE into standard form firstly.

$$y' - \frac{1}{3}\, y = \frac{1}{6}\, xy^4$$

The substitution has to be $u = y^{1-n} = y^{-3}$ and we multiply DE by $u' = -3y^{-4}$.

$$\begin{aligned}-3y^{-4}y' - (-3y^{-4})(\frac{1}{3}\, y) &= -3y^{-4}\, \frac{1}{6}\, xy^4 \\ \frac{du}{dx} + y^{-3} &= -\frac{3}{6}\, x \\ \frac{du}{dx} + u \cdot 1 &= -\frac{3}{6}\, x\end{aligned}$$

Now it is linear DE with $P(x) = 1$ and the integrating factor is $\mu(x) = e^x$.

$$e^x\, \frac{du}{dx} + e^x u = -e^x\, \frac{1}{2}\, x$$

It is expected that what is on the left side is $d/dx\ \mu(x)u$ and you can check that is is indeed.

$$\begin{aligned}\frac{d}{dx}\, e^x u &= -e^x\, \frac{1}{2}\, x \\ e^x u &= -\frac{1}{2} \int xe^x\, dx\end{aligned}$$

The integral $\int xe^x\, dx = xe^x - e^x$ have to be evaluated by parts.

From $(uv)' = u'v + uv' \implies u'v = (uv)' - uv' \implies \int u'v\, dx = uv - \int uv'\, dx$:

$$\int xe^x\ dx = \begin{vmatrix} v = x & v' = 1 \\ u' = e^x & u = e^x \end{vmatrix} = xe^x - \int e^x\ dx = xe^x - e^x + c$$

$$\begin{aligned}e^x u &= -\frac{1}{2}\, (xe^x - e^x) + c \\ u &= -\frac{1}{2}\, (x - 1) + ce^{-x}\end{aligned}$$

The general solution is then

$$y^{-3} = -\frac{1}{2}\, (x - 1) + ce^{-x}.$$

Applying initial value to determine c:

$$\left(\frac{1}{-2}\right)^3 = -\frac{1}{2}(0-1) + ce^0$$
$$-\frac{1}{2} - \frac{1}{8} = c \implies c = -\frac{5}{8}$$

The particular solution is then

$$\underline{\underline{y^{-3} = -\frac{1}{2}(x-1) - \frac{5}{8}e^{-x}}}$$

First order linear differential equations

Numerical method to solve DE

So far we have learnt how to solve by means of direction field and we learnt how to solve analytically some kinds of DE.

Not all differential equations can be solved analytically and we might obtain satisfying results by numerical solution. The most simple method is Euler's method.

This method is not accurate, but its principle is obvious. For its inaccuracy is not practically used. There exist software, which use Runge-Kutta methods.

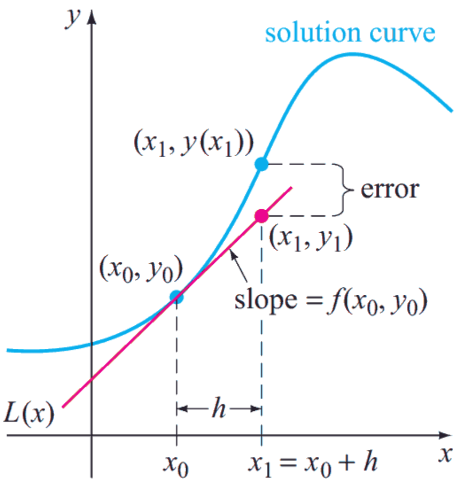


FIGURE 2.6.2 Approximating $y(x_1)$ using a tangent line

The curve in blue color is an unknown solution $f(x)$ to be found and h is a step width chosen. Since we are solving DE, we can also express from DE slope y' as a function of x, y.
From initial value $[x_0, y_0]$ we can move on the slope to $[x_1, y_1]$ and from that point continue to $[x_2, y_2]$, ...

In numerical solutions, it is usually desirable to construct a table in which all relevant computations are systematically recorded.

Example

Solve DE $y' = x^2 + y$ for initial condition y(0) = 1: find solution by Euler's method at

- x = 0.1,
- x = 0.2,
- x = 0.3

with steps

A. h = 0.1
B. h = 0.05

and compare with algebraic solution $y = 3e^x - x^2 - 2x - 2$.

A. Solved with step $h = \Delta x_n = 0.1$:

Numerical method					**Algebraic m.**
x_n	y_n	$y'(x_n, y_n)$	$\times\ \Delta x_n$	$=\ \Delta y_n$	y_n
0	**1**	1	0.1	0.1	
0.1	**1.1**	1.11	0.1	0.111	**1.1055**
0.2	**1.211**	1.251	0.1	0.1251	**1.2242**
0.3	**1.336**				**1.3596**

B. Solved with step $h = \Delta x_n = 0.05$:

Numerical method					**Algebraic m.**
x_n	y_n	$y'(x_n, y_n)$	$\times\ \Delta x_n$	$=\ \Delta y_n$	y_n

0	**1**	1	0.05	0.05	
0.05	**1.05**	1.0525	0.05	0.05263	
0.1	**1.1026**	1.1126	0.05	0.0556	**1.1055**
0.15	**1.1582**	1.1807	0.05	0.0590	
0.2	**1.2172**	1.2572	0.05	0.0629	**1.2242**
0.25	**1.2801**	1.3426	0.05	0.0671	
0.3	**1.3472**				**1.3596**

In general the accuracy of the method increases as the step h decreases. But not indefinitely: if the step is too small, rounding errors come into play.

Modeling with first order differential equations

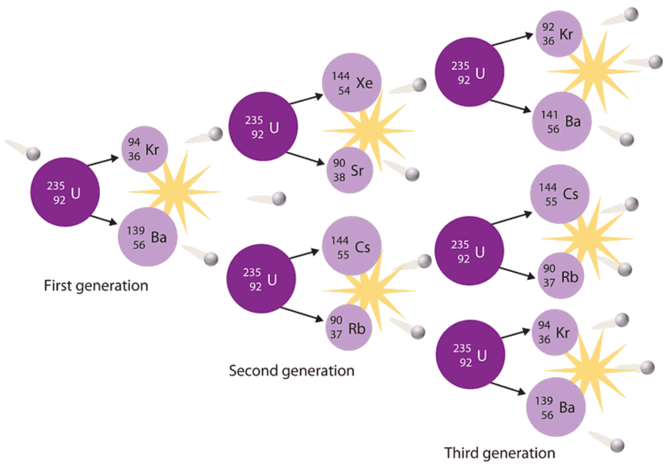
Let us consider first some common task. Like a number of products made in a factory. We assume that each day, the amount of newly produced goods is the same (constant). The result product from the factory is being accumulated, but **the change** of goods made at any day is zero.

But in some other problems **the change of amount grows/declines over time.**

Radioactive decay

Transmutation of radioactive particles depends on number of such particles. The number of observed transmutations is not constant in time, but (at given time) is e.g. 10 % of all radioactive particles. That also reminds so called half-life: for C^{14} is around 5600 years.

Carbon dating is used to determine approximate ages of fossilized matters. All living organism contain two isotopes of carbon: C^{12} and C^{14}. The first element is stable, the second is radioactive. Furthermore the ratio between them for *living* organism is constant within any known time epoch. However once the organism dies, the isotope C^{14} is being lost by radiation and is no longer being replaced.



Transmutation is caused by either decay or nuclear reaction. Many types of decay do cause transmutation of the decaying radioisotope.

So we have

$A(t)$	number of particles at given time
dA/dt	change of the number of particles through given timeframe

The increase or decrease of particles dA depends on number of all particles, on a constant k and on the time period (the longer we wait, the more particles transmute). Then the differential equations describing events is

$$dA = A(t)\, k\, dt,$$

where k is a constant binded with given general problem. We might think of k being a physical constant.

Newton's Law of cooling/warming

The speed of cooling/warming depends on distance of temperatures.

T_B	temperature of the immersed body
T_M	temperature of the medium (i.e. usually constant)
$(T_B - T_M)$	the higher the temperature is, the faster cooling/warming proceeds

$$dT_B = k(T_B - T_M)\, dt$$

When the DE is solved, the constant k has to be evaluated. That is usually done from temperature T_B read when the experiment starts (at $t = 0$) and from temperature observed later.

Mixtures

The mixing of two solutions when one solution is pumped into a tank can be readily solved by differential equations. Let us talk about brine (salt solution) in the tank, which is being released from the tank at a rate R_{OUT} while other solution is flowing into the tank at a rate R_{IN}.

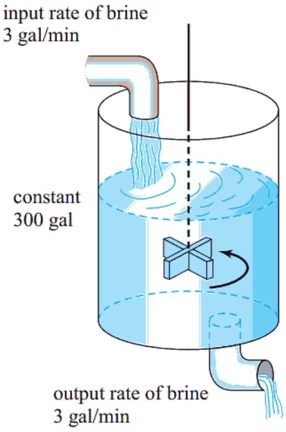


FIGURE 1.3.2 Mixing tank

A	amount of salt inside the tank (in grams)
R_{IN}	inflow: salt entering the tank (in grams) during one time unit
R_{OUT}	outflow: salt being released from the tank (in grams) during one time unit

The actual change of the amount of salt in grams in the container depends on salt flowing to and from the container:

$$dA = (R_{IN} - R_{OUT})\, dt$$

Draining a tank

We know from hydrodynamics that the water speed at outflow is $v = \sqrt{2gh}$. We want to analyze the height h of the water in given time t (the purpose might be to know volume of remaining water in the tank).

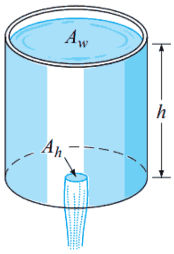


FIGURE 1.3.3 Draining tank

The released amount (volume) of water depends on speed, time and cross-section of the outflow pipe:

$$dV = -A_h v\, dt$$
$$dV = -A_h \sqrt{2gh}\, dt$$

Since volume $V = A_w h$, where A_w is constant, then $dV = A_w\, dh$:

$$A_w\, dh = -A_h \sqrt{2gh}\, dt \qquad \text{or} \qquad \frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Example

Half-life of C^{14} is 5600 years. The ruins of a town were evaluated as from 7000 BC. How much of carbon remains during observation?

9000 years ago the ruins contained 100 %. After 5600 years they contained 50 %. We might guesstimate, that today it would be around 35 %. But let us do rather the math.

Because we are talking about decay, it is good practice to place minus sign into DE. It does not affect the solution (the constant k would change its sign).

$$
\begin{aligned}
dA &= -A\,k\,dt \\
\frac{1}{A}\,dA &= -k\,dt \\
\log A &= -kt + c \\
A &= e^{-kt+c}
\end{aligned}
$$

So we have solved the DE in advance, but there are so many letters to deal with. Variable t is a time, we will substitute time in years. Variable $A(t)$ is amount of particles, in our case, it is 100 % at $t = 0$ and 50 % after 5600 years. The constant c will be determined from initial value and constant k (for C^{14}) as well. When both k, c are determined, we have particular solution which fits to the case and can be used to answer the question from the beginning.

For $A(t = 0) = 100\ \%$ and $A(t = 5600) = 50\ \%$:

$$
\left.
\begin{aligned}
100 &= e^{-k\cdot 0+c} \implies c = 4.605 \\
50 &= e^{-k\cdot 5600+4.605} \implies k = 1.237 \times 10^{-4}
\end{aligned}
\right\}
\qquad \underline{\underline{A = e^{-1.237\times 10^{-4}t+4.605}}}
$$

Substituting $t = 9000$ years into solution brings $A(t = 9000) = 32.8\ \%$. So, in the time of observation, there remained 32.8 % of carbon isotope.

Note: we could also consider $A(t = 0) = 1$ and $A(t = 5600) = 0.5$ instead of messing with per cents. In such case the solution is $A = e^{-1.237\times 10^{-4}t}$.

Example

The population of colony doubles in 50 days. In how many days will the population triple?

First, it is a good idea to make a guesstimate by intuition. That is useful to test the final result for correctness.

If we assign A to the amount of population, then we may form differential equation

$$
\begin{aligned}
dA &= k \cdot A \cdot dt \qquad \text{or} \\
\frac{dA}{dt} &= kA
\end{aligned}
$$

It says that the longer we wait, the larger the colony becomes. And the increase depends on the actual size of the colony: regardless the colony has one or 200 members, the increase dA will double, so it has to depend on A. There is also a constant k which has to be determined.

$$
\begin{aligned}
dA\,\frac{1}{A} &= k\,dt \\
\log A &= kt + c \\
A &= e^{kt+c}
\end{aligned}
\tag{23}
$$

We have to determine values of two constant k and c by means of initial values (IV). One IV is at the time of 50 days, the other at the time of beginning.

$$
\begin{aligned}
t = 0 &: A = 1 \\
t = 50 &: A = 2
\end{aligned}
$$

Substituting IV into (23) we make two equations which determine values of k and c and the particular solution can be formed:

$$
\underline{A = e^{0.01386t}} \tag{24}
$$

From the solution (24) it is easy task to get answer to initial question:

$$
\left.
\begin{aligned}
t = 0 &: A = 1 \\
t = ? &: A = 3
\end{aligned}
\right\}
\quad 3 = e^{0.01386t} \implies \underline{\underline{t = 79 \text{ days}}}
$$

Example

If 1.7 % of a substance decomposes in 50 years, A) what percentage of the substance will remain after 100 % years? B) How many years will be required for 10 % to decompose?

One has to realize quickly that such task deals with half-time of particles. There is a (yet unknown) time in which the amount A of particles always decreases to half: if the half-time is 1 day, each day the change dA will be half of A.

$$\begin{aligned} dA &= -k \cdot A \cdot dt \qquad \text{or} \\ \frac{dA}{dt} &= -kA \end{aligned}$$

The minus sign in above equation is less of importance. It expresses the fact that the amount of particles is decreasing in the time. Even if we omit minus sign, the equation will still work, only the constant k will be found negative.

$$A = e^{-kt+c}$$

$$\left.\begin{aligned} t = 0: A = 100~\% \\ t = 50: A = 98.3~\% \end{aligned}\right\} \quad \begin{aligned} 100 &= e^{-k\cdot 0+c} \implies c = 4.6052 \\ 98.3 &= e^{-k\cdot 50+4.6052} \implies k = 3.435 \cdot 10^{-4} \end{aligned}$$

The particular solution for given case is

$$\underline{A = e^{-\cdot 3.435\cdot 10^{-4}\ t+4.6052}}$$

Using the above solution we find the answers to questions from beginning:

$$\begin{aligned} &\text{A) } \underline{\underline{96.6~\%}} \\ &\text{B) } 90 = e^{-\cdot 3.435\cdot 10^{-4}\ t+4.6052} \implies \underline{\underline{t = 307 \text{ years}}} \end{aligned}$$

Example

A body whose temperature is 180 °C is immersed into liquid which is kept at a constant temperature of 60 °C. In one minute the temperature of the body decreases to 120 °C. How long will it take for the body to decrease to 90 °?

Let us assign T_B to temperature of the body, T_M to temperature of the medium. Then we can describe the problem by Newton's Law of Cooling/Warming. The temperature of the body is decreasing. The longer we wait, the more the temperature will decrease. The colder is the medium, the more the temperature will decrease:

$$\begin{aligned} dT_B &= -k(T_B - T_M)\,dt \qquad \text{or} \\ \frac{dT_B}{dt} &= -k(T_B - T_M). \end{aligned}$$

In above DE k is a parameter to be found. The minus sign is to depict that the temperature is decreasing. But even with positive sign the equation is going to work, only the constant k is going to have negative sign.

Let us solve the DE, $T_B(t)$ is unknown value which we are after and T_M is a constant only (in our case 60 °C).

$$\begin{aligned} dT_B \frac{1}{T_B - T_M} &= -k\,dt \\ \log(T_B - T_M) &= -kt + c \\ T_B &= T_M + e^{-kt+c} \end{aligned}$$

We have to solve constants k and c from initial values:

$$\begin{aligned} t = 0: T_B = 180\,^o\text{C} &\implies c = 4.787 \\ t = 1: T_B = 120\,^o\text{C} &\implies k = 0.693 \end{aligned}$$

Then we have $T_B = T_M + e^{-0.693t+4.787}$ and from this function we can express that answer to the question is that the body will decrease to 90 °C in two minutes.

Example

A dead body was found within a closed room of a house where the temperature was a constant 70 °F. At the time of discovery the core temperature of the body was determined to be 85 °F. One hour later a second measurement showed that the core temperature of the body was 80 °F. Assume that the time of death corresponds to t=0 and that the core temperature at that time was 98.6 °F. Determine how many hours elapsed before the body was found.

The room temperature is a constant $T_M = 70$. The body temperature $T_B(t)$ depends on time.

The room temperature	$T_M = 70\ ^o\mathrm{F}$	
The body temperature	$T_B = 98.6\ ^o\mathrm{F}$	$t = 0$
At unknown time Δ temperature observed	$T_B = 85\ ^o\mathrm{F}$	$t = \Delta$
One hour later the temperature observed	$T_B = 80\ ^o\mathrm{F}$	$t = \Delta + 1$ hour

$$\begin{aligned} dT_B &= -k(T_B - T_M)\, dt \\ \frac{dT_B}{dt} &= -k(T_B - T_M) \\ \frac{dT_B}{dt} &= -k(T_B - 70) \\ \frac{1}{T_B - 70}\, dT_B &= -k\, dt \\ \log T_B - 70 &= -kt + c \\ T_B &= 70 + e^{-kt+c} \end{aligned}$$

Now we have to use initial values to determine value of k, c and especially Δ.

From $t = 0,\ T_B = 98.6 \implies c = 3.353$.

From $t = \Delta,\ T_B = 85$ and $t = (\Delta + 1),\ T_B = 80 \implies \Delta = 1.6$ hour.

Example

A tank contains 200 liters of fluid in which 30 grams of salt are dissolved. Pure water is then pumped into the tank at a rate of 4 l/min; the well-mixed solution is pumped out at the same rate. Find the number A(t) of grams of salt in the tank at time t.

If $A(t)$ is an actual amount of salt in grams in the tank, then $dA = (R_{IN} - R_{OUT})\, dt$, where R_{IN} is a rate of incoming salt and R_{OUT} is rate of salt being released. In our case R_{IN} is zero. The outflow of salt at any time is $R_{OUT} = \frac{4}{200} A(t)$ in grams per second.

So we have differential equation

$$\begin{aligned} dA &= (-\frac{4}{200} A)\, dt \\ dA &= (-\frac{1}{50} A)\, dt \\ -50\, dA\, \frac{1}{A} &= dt \\ -50 \log A &= t + c \implies c = -170 \\ \log A &= -\frac{1}{50}(t + c) \end{aligned}$$

And the answer is

$$\underline{\underline{A(t) = 30e^{-1/50t}}}.$$

Example

A tank contains 200 liters of fluid in which 30 grams of salt are dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 4 l/min; the well-mixed solution is pumped out at the same rate. Find the number A(t) of grams of salt in the tank at time t.

What was discussed in the last example can be applied here. In addition $R_{IN} = 4$ grams per second enters the tank.

So we have differential equation

$$dA = (4 - \frac{4}{200}A)\,dt$$
$$dA = (4 - \frac{1}{50}A)\,dt$$
$$\frac{1}{4 - \frac{1}{50}A}\,dA = dt$$
$$-50\log(4 - \frac{1}{50}A) = t + c \implies c = -61.189$$
$$\log(4 - \frac{1}{50}A) = -\frac{1}{50}(t + c)$$
$$4 - \frac{1}{50}A = e^{-1/50(t+c)}$$
$$4 - \frac{1}{50}A = 3.4e^{-1/50t}$$

And the answer is

$$\underline{\underline{A(t) = 200 - 170e^{-1/50t}}}.$$

Example

A tank of 100 l contains 30 g of salt. Fresh water comes at 3 l/min and flows out at the same rate. Find the salt content after 10 minutes.

We know that the change of amount of salt $A(t)$ can be described by DE

$$dA = (R_{IN} - R_{OUT})\,dt$$

where R_{IN} is a salt in grams on inflow and R_{OUT} is a salt in grams on outflow. Now we have to ask how much is R_{IN} and how much is R_{OUT}. Since fresh water is coming into the tank, $R_{IN} = 0$.

In one minute, 3 from 100 litres are released, i.e. $\frac{3}{100}A(t)$ in grams of salt are released every minute. So, $R_{OUT} = \frac{3}{100}A(t)$

Let us make the differential equation:

$$dA = (0 - \frac{3}{100}A(t))\,dt$$
$$\frac{1}{A}\,dA = -\frac{3}{100}\,dt$$
$$\log|A| = -\frac{3}{100}t + c$$
$$A(t) = e^{-3/100\cdot t + c}$$

The solution is here, we have to find constant c. That will be done from initial value at the time $t = 0: \; A(t = 0) = 30$ grams.

$$30 = e^{-3/100\cdot 0 + c} \implies c = 3.401 \implies$$
$$\implies A(t) = e^{-3/100\cdot t + 3.401}$$

The question from the beginning asks for salt content after 10 minutes:

$$\underline{\underline{A(t) = e^{-3/100\cdot 10 + 3.401} = 22.22\text{ g}}}$$

Example

A tank holds 300 l of liquid in which is 50 g of salt. Another solution is pumped into tank at the rate of 3 l/min, its concentration is 2 g/l.

The solution is pumped out at the rate of 2 l/min.

Describe the problem by DE, if $A(t)$ is amount of salt in grams in the container at any time.

Again we have to determine R_{IN}, which is inflow of salt in grams per time unit. In this case $R_{IN} = 3 \cdot 2 = 6$ g/min.

Again we have to determine R_{OUT}, which is the outflow of salt in grams per time unit. In this case, in one minute, $\frac{2}{V}\,A(t)$ grams are flowing out. Since 3 l/s are on inflow and only 2/l on outflow, the volume is now function of time: $V(t) = 300 + 1 \cdot t$.

$$dA = \left(3 \cdot 2 - \frac{2}{V(t)}\,A(t)\right)\,dt = \left(6 - \frac{2}{300+t}\,A(t)\right)\,dt$$

or

$$\underline{\underline{\frac{dA}{dt} = 6 - \frac{2}{300+t}\,A(t)}}$$

Example

The case like the above case, but the water flows out at 3.5 l/min.

Not much has to be changed compared to the previous example. Only this time the volume is decreasing: $V(t) = 300 - 0.5t$. Then

$$dA = \left(3 \cdot 2 - \frac{3.5}{V(t)}\,A(t)\right)\,dt = \left(6 - \frac{3.5}{300-0.5t}\,A(t)\right)\,dt$$

or

$$\underline{\underline{\frac{dA}{dt} = 6 - \frac{7}{600-t}\,A(t)}}$$

Higher order differential equations

Examples:

$$\left.\begin{aligned} y'' + 3y' + y &= 0 \\ y''' + 2y'' - y' - 2y &= 0 \\ x^2y'' + xy' - y &= 0 \end{aligned}\right\} \quad \text{homogeneous linear DE}$$

$$\left.\begin{aligned} y'' + 3y' + y &= 4 \\ y''' + 2y'' - y' - 2y &= 12e^x \\ x^2y'' + xy' - y &= 1 \end{aligned}\right\} \quad \text{nonhomogeneous linear DE}$$

The general description of higher order linear differential equations is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{25}$$

Homogeneous DE, which has zero member $g(x)$ on the right side, is associated with non-homogeneous DE.

Initial conditions

Example:

$$\begin{aligned} y(1) &= 7 \\ y'(1) &= 0 \\ y''(1) &= 0 \end{aligned}$$

For first order DE we had only one pair of initial values. For higher order DE, initial conditions have to pass given point x_0 and fulfill prescribed values $y(x_0),\ y'(x_0),\ y''(x_0),\ \ldots$

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ y''(x_0) &= y_2 \\ &\vdots \\ y^{n-1}(x_0) &= y_{n-1} \end{aligned}$$

Existence of unique solution

If all coefficients $a_n(x)$ and $g(x)$ are continuous functions on an interval I containing x_0 and $a_n(x) \neq 0$ for every x on interval, then linear DE has one and only one solution $y = y(x)$, which satisfies set of initial conditions $y(x_0) = y,\ y'(x_0) = y_1,\ \ldots,\ y^{n-1}(x_0) = y_{n-1}$.

In many cases, knowing that a unique solution exists is more important than the solution itself.

Example

Differential equation $x^2y'' - 2xy' + 2y = 6$ has a solution $y = cx^2 + x + 3$ for initial values $y(0) = 3, y'(0) = 1$.

Let us observe $a_n(x)$, i.e. $a_2(x)$. Because $a_2(x)$ is zero for $x_0 = 0$, the solution is not unique for given initial values. We can test, that every c satisfies solution. There is no unique solution.

Example

Differential equation $(x-2)y'' + 3y = x$ with initial values $y(0) = 0, y'(0) = 1$.

We are able to find an interval $(-\infty; 2)$ where $a_2(x) \neq 0$. Since x_0 belongs to that interval where $a_2(x) \neq 0$, there exists unique solution then.

Example

Find the largest interval where differential equation $(t^2-1)y'' + 3ty' + \cos ty = t$ with $y(0)=4, y'(0)=5$ is guaranteed to have a unique solution.

We have to use *theorem of unique solution*. The theorem states that all coefficients $a_0(x),\ a_1(x),\ \ldots,\ a_n(x)$ have to be continuous on given interval. If $a_n(x)=0$ at a point within the interval, then continuity is broken.

Note: there is an analogy with equation $yx=1 \implies y=1/x$. If $x=0$ the value is not defined and the function is not continuous.

In our case for $x_0=0$ there exists an interval $\underline{\underline{(-1;1)}}$ where $a_0(x),\ a_1(x),\ a_2(x)$ are continuous and $a_n(x)\neq 0$ for any t from the interval.

Superposition principle

Let $y_1,\ y_2,\ \ldots,\ y_k$ be solution of the homogeneous n-th order DE. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

with arbitrary constants c_n is also solution.

Example

If e^{-2x} is a solution of given DE $y' + 2y' = 0$, then ce^{-2x} is also solution.

Example

If $y_1 = x^2$ and $y_2 = e^x$ are both solution of given DE, then $y = c_1 x^2 + c_2 e^x$ is also solution.

Linear dependence and independence

A set of functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ is **linearly dependent** on an interval I if we can find constants c_1, c_2, ..., c_n (not all set to zero) such, that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If the set is not linearly dependent on the interval, it is said to be linearly independent.

Example

$$\left.\begin{aligned} f_1(x) &= x \\ f_2(x) &= -3x \end{aligned}\right\} \quad \text{Can we find some constants such, that the combination is zero?}$$

$$\left.\begin{aligned} c_1 &= -3 \\ c_2 &= 1 \end{aligned}\right\} \quad f_1(x) \text{ and } f_2(x) \text{ are linearly dependent}$$

Example

$$\left.\begin{aligned} f_1(x) &= x \\ f_2(x) &= e^x \end{aligned}\right\} \quad \text{Can we find some constants such, that } c_1 f_1(x) + c_2 f_2(x) = 0 \text{ on an interval?}$$

Can not $\implies$ $f_1(x)$ and $f_2(x)$ are linearly independent.

Example

$$\left.\begin{aligned} f_1(x) &= 0 \\ f_2(x) &= \sin x \end{aligned}\right\} \quad \text{Can we find some constants such, that } c_1 \cdot 0 + c_2 \sin x = 0?$$

If $c_2 = 0$, then c_1 can be nonzero and $0 = 0$ for every x on interval. Thus $f_1(x)$ and $f_2(x)$ are linearly dependent.

Wronskian

When solving DE, **we are interested into linearly independent functions.** There is a mechanism how to resolve whether given functions are independent:

The Wronskian is a determinant, which consists of a set of functions $f_1(x),\ f_2(x),\ \ldots,\ f_n(x)$ which passes at least $n-1$ derivatives.

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{n-1}(x) & f_2^{(n-2)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

The set of functions $f_1,\ f_2,\ \ldots,\ f_n$ is linearly independent if $W(f_1,\ f_2,\ \ldots,\ f_n) \neq 0$ for every x in the interval.

Example

Test whether $f_1(x) = e^{3x}$ and $f_2(x) = e^{-3x}$ are linearly independent functions.

$$W(f_1, f_2) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = e^{3x}(-3e^{-3x}) - e^{-3x} \cdot 3e^{3x} = -3 - 3 = -6$$

$W \neq 0 \implies$ the two functions are linearly independent.

Solution of DE of higher order

Homogeneous DE

The homogeneous DE of order n has a fundamental set of linearly independent solutions $y_1,\ y_2,\ \ldots,\ y_n$.

General solution is then linear combination of solutions from the set

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Above function $\boldsymbol{y_c}$ is solution of homogeneous DE and is also called **complementary function**. The higher is the degree of DE, the more functions is involved. For example, if DE is of **2nd** order, then $y_c(x)$ consists of **two independent solutions.**

Nonhomogeneous DE

The nonhomogeneous DE has a solution

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a solution found for associated homogeneous DE (above) and $y_p(x)$ is a particular solution of the nonhomogeneous DE.

Example

$y'' + 4y' + 4y = 4x^2 + 6e^x$ is nonhomogeneous DE.

Associated homogeneous DE is $y'' + 4y' + 4y = 0$, which has a solution $y_c(x) = (c_1 + c_2 x)e^{-2x}$.

The general solution $y(x) = y_c(x) + y_p(x)$ involves solution of associated DE and also particular solution for given nonhomogeneous DE (see member $4x^2 + 6e^x$; it will be explained in other chapter where $y_p(x)$ comes from):

$$y = (c_1 + c_2 x)e^{-2x} + \qquad \text{complementary function}$$
$$+ x^2 - 2x + \frac{3}{2} + \frac{2}{3}e^x \qquad \text{particular solution}$$

Higher order linear differential equations

Solution of homogeneous DE

We will investigate two cases:

1. DE with **non-constant coefficients** $a_0(x),\ a_1(x),\ \ldots,\ a_n(x)$
   Example: $(2x^2+1)y''-4xy'+4y=0$ (coefficients are function of x) and
2. DE with **constant coefficients** $a_0,\ a_1,\ \ldots,\ a_n$
   Example: $y^{(4)}+2y''+y=0$.

In actual practice equations of the first type have solutions, which are usually not expressible in terms of elementary functions. And even if they are, it is extremely difficult to find them. We will show how to find the other solution $y_2()$ for DE of 2nd order assuming we have one solution $y_1()$ already.

If the coefficients are constant as in the second case, then the solution can be readily obtained.

Higher order linear differential equations

Solution by reduction of order for linear DE with non-constant coefficients

Example: we want to solve

$$x^2y'' + xy' - y = 0$$

Such DE might be difficult to solve, since it has non-constant coefficients. However, **sometimes it is possible to find at least one solution** (in above case $y_1(x) = x$) **and then use method described further to derive the second solution $y_2(x)$ from the first one.**

The method can be used for higher order DE and will be shown on 2nd order DE. The aim of this chapter is to bring a generalized convenient formula, which can be used to get $y_2(x)$ readily from $y_1(x)$.

The differential equation is linear but with non-constant coefficients. First, we bring DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ into standard form

$$y'' + P(x)y' + Q(x)y = 0 \tag{26}$$

Let us suppose that we have one solution already ($y_1(x)$). Either it is given or easy to find. We want to find the other solution $y_2(x)$ of given DE. Since $y_1(x)$ and $y_2(x)$ are linearly independent, we can express y_2 as

$$y_2(x) = u(x) \cdot y_1(x). \tag{27}$$

Let us differentiate $y_2(x)$ two times

$$\begin{aligned} y_2'(x) &= u'y_1 + uy_1' \\ y_2''(x) &= u''y_1 + u'y_1' + u'y_1 + uy_1'' = u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

and place the products into standard form (26) of DE:

$$[u''y_1 + 2u'y_1' + uy_1''] + [P(x)(u'y_1 + uy_1')] + [Q(x)uy_1] = 0 \tag{28}$$

$$u(y_1'' + Py_1' + Q(x)y_1) + u''y_1 + 2u'y_1' + P(x)u'y_1 = 0 \tag{29}$$

The content of parentheses in (29) is known to be zero, because y_1 is a solution of homogeneous DE (26).

$$\begin{aligned} y_1u'' + 2y_1'u' + P(x)y_1u' &= 0 \\ y_1u'' + u'(2y_1' + P(x)y_1) &= 0 \end{aligned} \tag{30}$$

Reminder: y_1 and y_1' are known, we are solving u. Let us use substitution $w = u'$:

$$y_1w' + w(2y_1' + P(x)y_1) = 0 \tag{31}$$

The last equation (31) can be solved by separating variables:

$$\begin{aligned} y_1\frac{dw}{dx} + w(2y_1' + P(x)y_1) &= 0 \\ dw\frac{1}{w} + dx(2\frac{y_1'}{y_1} + P(x)) &= 0 \\ \log w + 2\int\frac{y_1'}{y_1}dx + \int P(x)dx &= 0 \end{aligned} \tag{32}$$

The integral $\int(y'/y)dx$ might be something unexpected to solve, but is not difficult. Since $y' = (dy/dx)$

$$\int\frac{y'}{y}dx = \int\frac{dy}{dx}\frac{1}{y}dx = \int\frac{1}{y}dy = \log y.$$

Let us continue from (32):

$$\log(wy_1^2) = -\int P(x)dx + c$$
$$wy_1^2 = e^{-\int P(x)dx + c} =$$
$$= ce^{-\int P(x)dx}$$
$$w = c\,\frac{e^{-\int P(x)dx}}{y_1^2}$$
$$u = c_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} + c_2$$

We are looking for linear independent function and we will place arbitrary constant with y_1 and y_2. So c_1, c_2 are not much of our concern:

$$u(x) = \int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx \quad \text{and} \tag{33}$$
$$y_2(x) = u(x) \cdot y_1(x). \tag{34}$$

Equation (34) above is our expression (27) of $y_2(x)$ in terms of $u(x)$ and $y_1(x)$ from beginning of the lesson. Formula (33) is then an important equation expressing $u(x)$ in order to find the other solution $y_2(x)$.

Example

Solve $x^2y'' + xy' - y = 0$ if $y_1 = x$ is known to be first solution.

The *reduction of order* method is able to find $y_2(x)$ from first solution $y_1(x)$. First we need to bring DE into expected standard form:

$$y'' + \frac{x}{x^2}y' - \frac{1}{x^2}y = 0 \qquad x \neq 0$$
$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0 \qquad \frac{1}{x} = P(x)$$

Let us use $P(x)$ and $y_1(x)$ within (33) in order to determine $y_2(x)$ from (34) :

$$u(x) = \int \frac{e^{-\int 1/x\,dx}}{y_1^2(x)}\,dx = \int \frac{e^{-\log x}}{x^2}\,dx =$$
$$= \int \frac{-x}{x^2}\,dx = -\int x^{-3}dx = -\frac{1}{2}x^{-2}$$
$$y_2(x) = u(x) \cdot y_1(x) = -\frac{1}{2}x^{-2}x = -\frac{1}{2}x^{-1} = x^{-1}$$

Since we have both $y_1(x)$ and $y_2(x)$ we can collect complete complementary solution

$$\underline{\underline{y_c(x) = c_1x + c_2x^{-1}}}$$

Example

Solve DE $(2x^2+1)y'' - 4xy' + 4y = 0$ if $y_1(x) = x$ is known to be first solution.

This DE is linear with non-constant coefficients and is hard to solve. However because we know the first solution $y_1(x)$, we can use above method to find $y_2(x)$ and complete the solution.

The method was derived from DE in standard form, let us bring DE into standard form then:

$$
\begin{aligned}
&y'' - \frac{4x}{(2x^2+1)}\,y' + \frac{4}{(2x^2+1)}\,y = 0 \\
&u = \int \frac{e^{-\int -4x/(2x^2+1)\,dx}}{y_1^2}\,dx = \int \frac{e^{\log(2x^2+1)}}{x^2}\,dx = \int \frac{2x^2+1}{x^2}\,dx \\
&u = \int 2dx + \int \frac{1}{x^2}\,dx = 2x - x^{-1} \\
&y_2(x) = x\cdot(2x - x^{-1}) = 2x^2 - 1 \\
&\underline{\underline{y_c(x) = c_1 x + c_2(2x^2-1)}}
\end{aligned}
$$

Example

Use the *reduction of order* method to find general solution of DE $x^2y'' - xy' + y = 0$, $y_1 = x$.

$$y'' - \frac{x}{x^2}\,y' + \frac{1}{x^2}\,y = 0$$

We have to find $y_2(x) = u(x)\cdot y_1(x)$. From above DE, which was brought into standard form we will take $P(x) = -1/x$.

$$
\begin{aligned}
&u(x) = \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx = \int \frac{e^{-\int -1/x\,dx}}{x^2}\,dx = \int \frac{e^{\log x}}{x^2}\,dx = \int \frac{1}{x}\,dx = \log|x| \\
&y_2(x) = u(x)\cdot y_1(x) = \log|x|x \\
&\underline{\underline{y_c(x) = c_1 x + c_2 \log|x|\cdot x}}
\end{aligned}
$$

Example

Use the *reduction of order* method to find general solution of DE $y'' + 9y = 0$, $y_1 = \sin 3x$.

$$y'' - \frac{x}{x^2}\,y' + \frac{1}{x^2}\,y = 0$$

We have to find $y_2(x) = u(x)\cdot y_1(x)$. From above DE, which was brought into standard form we will take $P(x) = 0$, becaue member with y' is missing.

$$
\begin{aligned}
&u(x) = \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx = \int \frac{1}{\sin^2 3x}\,dx = \frac{1}{3\tan 3x} \\
&y_2(x) = u(x)\cdot y_1(x) = \frac{1}{3\tan 3x}\cdot \sin 3x = -\frac{1}{3}\frac{\cos 3x}{\sin 3x}\sin 3x = -\frac{1}{3}\cos 3x \\
&\underline{\underline{y_c(x) = c_1 \sin 3x + c_2 \cos 3x}}
\end{aligned}
$$

Example

Use the *reduction of order* method to find general solution of DE $x^2y'' + xy' - 4y = x^3$, $y_1 = x^2$.

Note: you might be not able to understand this example, because although the example is solved by the method explained above, it also expects some knowledge from chapters behind.

You may wonder where the solution x^2 comes from. But if we study the associated homogeneous DE, it has to be not too difficult to try a few simple solutions like $y = x$, $y = c_1$, $y = x^2$ and test, whether they are the solution or not. Then we can use the method to derive the next solution y_2.

Note: it can be also observed that the DE can be solved as a Cauchy-Euler DE.

To use the *reduction of order* method we have to bring the DE into standard form to find $P(x)$.

$$y'' + \frac{1}{x}\,y' - \frac{4}{x^2}\,y = x \implies P(x) = \frac{1}{x}$$

Now we can evaluate $u(x)$, once we have $u(x)$ we use it to multiply y_1 in order to get the other solution $y_2 = u(x) \cdot y_1(x)$.

$$u(x) = \int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\, dx = \int \frac{e^{-\int 1/x\, dx}}{x^4}\, dx = \int \frac{1/x}{x^4}\, dx = \int x^{-5}\, dx = -\frac{1}{4}\, x^{-4}$$

The constant $-\frac{1}{4}$ can be lost, because an arbitrary constant c_2 will be included anyway.

$$y_2 = u(x) \cdot y_1(x) = -\frac{1}{4}\, x^{-4} \cdot x^2 = -\frac{1}{4}\, x^{-2}$$
$$y = c_1 x^2 + c_2 x^{-2} + y_p$$

The task remains to find y_p since DE given to solve is nonhomogeneous. That can be done by method of undetermined coefficients. We might assume that the particular solution is going to be in the form of $y_p = Ax^2 + Bx + C$, then evaluate $y_p',\ y_p''$, substitute to DE in order to find coefficients $A,\ B,\ C$. But you will observe soon that it does not lead to a solution. That is because we deal with the second type (the member $g(x)$ contains a member from y_c). The next attempt to assume $y_p = Ax^3 + Bx^2 + Cx + D$ brings the constants $A = 1/5,\ C = 0,\ D = 0$ and then the general solution is:

$$\underline{\underline{y = c_1 x^2 + c_2 x^{-2} + \frac{1}{5}\, x^3}}.$$

Higher order linear differential equations

Homogeneous DE with constant coefficients

Example: we want to solve $y''' + 2y'' - y' - 2y = 0$. We will show, that it leads to solution of polynom $m^3 + 2m^2 - m - 2 = 0$ and the solution involves members $y = e^{mx}$.

We have learnt from linear DE of first order that the solution contains member e^{mx}. Without going into details, let us say that the possible solution of differential equation

$$a_n y^{(n)} + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 y = 0 \tag{35}$$

is

$$y = e^{mx}. \tag{36}$$

For what value of m will (36) be a solution of (35)?

$$a_n \frac{d^n}{dx^n} e^{mx} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} e^{mx} + \ldots a_1 \frac{d}{dx} e^{mx} + a_0 e^{mx} = 0. \tag{37}$$

Since k-th derivative of $e^{mx} = m^k e^{mx}$, we may rewrite (37) to

$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0 \tag{38}$$

Let us divide (38) by e^{mx} to obtain **auxiliary (characteristic)** equation:

$$a_n m^n + a^{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0. \tag{39}$$

The task diminished to an algebra task of finding roots m. We have to deal with the facts that

1. We can find roots of quadratic equation $ax^2 + bx + c = 0$, but it is not a simple task to find roots of higher order polynomial equations.
2. There are 3 possibilities what kind of roots might occur:
   A. **Roots are distinct and real** (e.g. $m_1 = 1$, $m_2 = 3$)
   B. Some of the **roots repeat** (e.g. $m_1 = 2$ and $m_2 = 2$)
   C. **Roots are imaginary** (e.g. $m_1 = 11 + 2i$, $m_2 = 11 - 2i$)
   The imaginary roots must occur in conjugate pairs $\alpha \pm i\beta$

A. Roots of auxiliary equation are real and distinct

The solution consists of functions $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$, $\ldots$, $y_n = e^{m_n x}$ or

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Because the roots do not repeat, is is given that above terms are linearly independent.

Example

Find the general solution of $y''' + 2y'' - y' - 2y = 0$.

We construct the auxiliary equation:

$$m^3 + 2m^2 - m - 2 = 0 \implies m = \{-2, 1, -1\}.$$

The solution is

$$\underline{\underline{y_c = c_1 e^{-2x} + c_2 e^x + c_3 e^{-x}}}.$$

Example

Find particular solution $y(x)$ of $y'' - 3y' + 2y = 0$ for which $y_c(0) = 1$ and $y_c'(0) = 0$.

Auxiliary form is

$$m^2 - 3m + 2 = 0 \implies m = \{1, 2\}.$$
$$y_c(x) = c_1 e^x + c_2 e^{2x}$$
$$y_c'(x) = c_1 e^x + 2c_2 e^{2x}$$

Substitute initial values:

$$\left.\begin{aligned} 1 &= c_1 e^0 + c_2 e^0 \\ 0 &= c_1 e^0 + 2c_2 e^0 \end{aligned}\right\} \begin{aligned} c_1 &= 2 \\ c_2 &= -1 \end{aligned}$$

Substituting constants into complementaty function y_c we have particular solution for given initial values:

$$\underline{\underline{y_c = 2e^x - e^{2x}}}.$$

B. Some roots are multiple

Even if the roots repeat, the members of solution can not repeat. Because we expect only functions, which are linearly independent. It can be shown (*reduction of order method*), that if roots are $m_1 = m_2$, the solution is $e^{m_1 x} + xe^{m_2 x}$.

In general it can be shown that if auxiliary equation has a root $m = a$ which repeats n times, then the general solution is

$$y_c = (c_1 + c_2 x + c_3 x^2 + \ldots c_n x^n)e^{ax}.$$

Example

Find the general solution of $y(x)$ of $y^{(4)} - 3y'' + 2y' = 0$.

$$m^4 - 3m^2 + 2m = 0 \implies m = \{0, 1, 1, -2\}$$
$$y_c = c_1 e^{0x} + c_2 e^x + c_3 x e^x + c_4 e^{-2x}$$
$$\underline{\underline{y_c = c_1 + c_2 e^x + c_3 x e^x + c_4 e^{-2x}}}$$

C. Roots are imaginary

The imaginary root must occur in conjugate pairs: if $\alpha + \beta$ is one root, another root must be $\alpha - i\beta$.

If roots are complex $\alpha \pm i\beta$ then the solution is

$$y_c = c_1 e^{(\alpha - i\beta)x} + c_2 e^{(\alpha + i\beta)x}.$$

We usually do not want to work with complex numbers. The solution above can be written also as

$$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

Example

Find the general solution of $y^{(4)} + 2y'' + y = 0$.

$$m^4 + 2m^2 + 1 = 0 \quad \text{or} \quad (m^2 + 1)^2 = 0 \implies m = \{i, i, -i, -i\}$$

Since complex number has a form $\alpha \pm i\beta$, we have $\beta = 1$ and the number repeats.

$$y_c = c_1 e^{ix} + c_2^{-ix} + c_3 e^{ix} x + c_4 e^{-ix} x$$
$$\underline{\underline{y_c = (c_1 + c_3) x e^{ix} + (c_2 + c_4) x e^{-ix}}}$$

or

$$y_c = e^0 (c_1 \cos x + c_2 \sin x) + e^0 x (c_3 \cos x + c_4 \sin x)$$
$$\underline{\underline{y_c = (c_1 + c_3 x) \cos x + (c_2 + c_4 x) \sin x}}$$

Problem solving

Example

Solve $y'' + 4y' + 4y = 0$.

$$m^2 + 4m + 4 = 0 \implies m = \{-2, -2\}$$
$$\underline{\underline{y = c_1 e^{-2x} + c_2 x e^{-2x}}}$$

Example

Solve $y^{(4)} - 2y'' = 0$.

$$m^4 - 2m^2 = 0$$
$$(m - \sqrt{2})^2 m^2 = 0 \implies m = \{-\sqrt{2}, \sqrt{2}, 0, 0\}$$
$$y_c = c_1 e^{-\sqrt{2}x} + c_2 e^{\sqrt{2}x} + c_3 e^0 + c_4 x e^0$$
$$\underline{\underline{y_c = c_1 e^{-\sqrt{2}x} + c_2 e^{\sqrt{2}x} + c_3 + c_4 x}}$$

Example

Solve $y'' - 2y' + 5y = 0$.

$$m^2 - 2m + 5 = 0 \implies m = \{1 + 2i, 1 - 2i\} \quad (\alpha = 1, \beta = 2)$$
$$y_c = c_1 e^{(1+2i)x} + c_2 e^{(1-2i)x}$$
$$\underline{\underline{y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)}}$$

Higher order linear differential equations

Undetermined coefficients

So far we have dealt with homogeneous DE, e.g.

$$y'' - 3y' + 2y = 0$$

When we found the complementary function $y_c(x)$, we got the general solution $y(x)$.

$$y(x) = y_c(x) = c_1 e^x + c_2 e^{2x}$$

Now we have to deal with nonhomogeneous DE, e.g.

$$y'' - 3y' + 2y = x + \sin x$$

The DE above has right side of $g(x) = x + \sin x$ or $g(x) \neq 0 \implies$ DE is not homogeneous.

In this case the general solution consists of complementary function $y_c(x)$ and also of particular solution $y_p(x)$. We showed how to find $y_c(x)$ already in previous lesson. **Now remains the problem of finding $y_p(x)$.**

Method of indeterminate coefficients (superposition and annihilator approach)

There are restrictions to use the methods. The function $g(x)$ has to have members such is a, x^k, e^{ax}, $\sin ax$, $\cos ax$ and their combinations. These **functions have finite number of linearly independent derivatives.**

For example x^3 has a finite number of linearly independent derivatives: $3x^2$, $6x$, 6. The same can not be said about x^{-1}, which has infinite numbers of linearly independent derivatives.

Methods are described further:

- Superposition approach
- Annihilator approach

Higher order linear differential equations

Undetermined coefficients—Superposition approach

The method deals with the task of **finding particular solution $y_p(x)$** once we have complementary function $y_c(x)$. General restrictions and type of DE which are being solved are covered in a overview of Undetermined coefficients method. We have to examine right side $g(x)$ of DE and compare its members against members within $y_c(x)$:

1. **No member in $g(x)$ is the same as a member of $y_c(x)$.**
   The particular solution then involves all members of $g(x)$ and its derivatives.
2. **$g(x)$ contains a function which is x^k times a member within $y_c(x)$.**
   In such case particular solution $y_p(x)$ has to be based also on that member but multiplied by x^{k+r}, where value of r (typically 1 or 2) has to be decided during solution.

Finally the arbitrary constants for the above members (forming $y_p(x)$) have to be found and that will be explained rather on examples.

Example

Find the general solution of $y'' + 4y' + 4y = 4x^2 + 6e^x$

Short observation tells us it is linear DE with constant coefficients so it will be simple task to find $y_c(x)$ by means of auxiliary equation. It is also seen that there are members on the right side so it is nonhomogeneous DE and we have to find y_p using method of undeterminated coefficients.

Let us construct auxiliary equation to find $y_c(x)$:

$$\begin{aligned} &m^2 + 4m + 4 = 0 \\ &(m+2)(m+2) = 0 \implies m = \{-2, -2\} \\ &\underline{y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}} \end{aligned}$$

Note: no member observed within $g(x)$ appears in complementary function. Therefore the solution of $y_p(x)$ is simple and straight-forward. Particular solution $y_p(x)$ will consist of x^2, e^x and its derivatives (x, 1):

$$y_p(x) = Ax^2 + Be^x + Cx + D$$

We have to find proper A, B, C, D to complete particular solution. The coefficients will be obvious when we use the particular solution $y_p(x)$ within DE (we know that $y_p(x)$ is a solution of DE so there is nothing wrong with that). Let us prepare its derivatives and let us feed them into DE then.

$$\begin{aligned} y_p'(x) &= 2Ax + Be^x + C \\ y_p''(x) &= 2A + Be^x \end{aligned}$$

Now substitute $y_p(x)$, $y_p'(x)$ and $y_p''(x)$ into DE

$$\begin{array}{cccccl} y'' & + & 4y' & + & 4y & = 4x^2 + 6e^x \\ (2A + Be^x) & + & 4(2Ax + Be^x + C) & + & 4(Ax^2 + Be^x + Cx + D) & = 4x^2 + 6e^x \end{array}$$

Now let us collect the same members of $y_p(x)$ together to find out values of constants:

$$x^2(4A) + e^x(B + 4B + 4B) + x(8A + 4C) + 1(2A + 4C + 4D) = 4x^2 + 6e^x$$

$$\left.\begin{aligned} &4A = 4 \\ &9B = 6 \\ &8A + 4C = 0 \\ &2A + 4C + 4D = 0 \end{aligned}\right\} \implies \begin{aligned} &A = 1 \\ &B = 2/3 \\ &C = -2 \\ &D = 3/2 \end{aligned}$$

General solution $y(x) = y_c(x) + y_p(x)$:

$$y(x) = c_1 e^{-2x} + c_2 e^{-2x} x + x^2 - 2x + \frac{3}{2} + \frac{2}{3} e^x .$$

Example

Solve DE $y'' + 3y' + 2y = \sin x$.

$$\begin{aligned} & m^2 + 3m + 2 = 0 \quad \Longrightarrow \quad m = \{-1, -2\} \\ & y_c(x) = c_1 e^{-x} + c_2 e^{-2x} \end{aligned}$$

Particular solution $y_p(x)$ contains $\sin x$ and its derivatives:

$$y_p(x) = A \sin x + B \cos x$$

Now the task is to find A and B which satisfy DE:

$$\begin{aligned} y_p'(x) &= A \cos x - B \sin x \\ y_p''(x) &= -A \sin x - B \cos x \end{aligned}$$

Let us substitute y_p, y_p' and y_p'' into DE:

$$(-A \sin x - B \cos x) + 3(A \cos x - B \sin x) + 2(A \sin x + B \cos x) = \sin x$$

Now we collect all $\cos x$ and all $\sin x$ together to determine A and B:

$$\sin x(-A - 3B + 2A) + \cos x(-B + 3A + 2B) = \sin x \quad \Longrightarrow$$
$$\Longrightarrow \begin{cases} (-3B + A) = 1 \\ (3A + B) = 0 \end{cases} \Longrightarrow \quad \begin{aligned} A &= 1/10 \\ B &= -3/10 \end{aligned}$$

The general solution is

$$y_c(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{10} \sin x - \frac{3}{10} \cos x$$

Example

Solve DE $y'' - 3y' + 2y = 2x^2 + 3e^{2x}$.

$$y_c(x) = c_1 e^x + c_2 e^{2x}$$

Particular solution $y_p(x)$ has to involve all members from $g(x)$ and its derivatives. The issue is that member e^{2x} is already included within $y_c(x)$ and has arbitrary constant. Therefore **we have to involve also xe^{2x}.** If we do not include that member, De^{2x} will cancel each other within the left hand side, because it is known to be one of the solutions of associated homogeneous DE. Then it would be not possible to express D. So now we have the *case 2* from lesson ($e^{2x} = x^0 e^{2x}$).

$$y_p(x) = Ax^2 + Bx + C + De^{2x} + Exe^{2x}$$

Simetimes the first order still does not lead to solution and higher orders might be required (x^2, ...).

$$\begin{aligned} y_p(x) &= Ax^2 + Bx + C + De^{2x} + Exe^{2x} \\ y_p'(x) &= 2Ax + B + Ee^{2x} + 2Exe^{2x} \\ y_p''(x) &= 2A + 2Ee^{2x} + 2Ee^{2x} + 4Exe^{2x} = \\ &= 2A + (2E + 2E)e^{2x} + 4Exe^{2x} \end{aligned}$$

After substituting y_p, y_p' and y_p'' into DE:

$$\begin{array}{rl} 2A + 4Ee^{2x} + 4Exe^{2x} & - \\ -3(2Ax + B + Ee^{2x} + 2Exe^{2x}) & + \\ +2(Ax^2 + Bx + C + Exe^{2x}) & = 2x^2 + 3e^{2x} \end{array}$$

Now we collect all members of the same kind together:

$$\begin{array}{rl} e^{2x}(4E - 3E) & + \\ +xe^{2x}(4E - 6E + 2E) & + \\ +2Ax^2 & + \\ +x(-6A + 2B) & + \\ +1(2A - 3B + 2C) & = 2x^2 + 3e^{2x} \end{array} \implies \left.\begin{array}{r} 2A = 2 \\ E = 3 \\ -6A + 2B = 0 \\ 2A - 3B + 2C = 0 \end{array}\right\} \implies \begin{array}{l} A = 1 \\ B = 3 \\ C = 7/2 \\ E = 3 \end{array}$$

The general solution is

$$\underline{\underline{y_c(x) = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 3xe^{2x}}}$$

Example

Solve $y'' + 3y' + 2y = 12e^x$.

$$\begin{array}{l} m^2 + 3m + 2 = 0 \implies m = \{-1, -2\} \\ \underline{y_c = c_1 e^{-x} + c_2 e^{-2x}} \\ y_p = Ae^x \\ y_p' = Ae^x \\ y_p'' = Ae^x \\ Ae^x + 3Ae^x + 2Ae^x = 12e^x \\ e^x(6A) = 12e^x \\ 6A = 12 \implies A = 2 \\ \underline{y_p = 2e^x} \\ \underline{\underline{y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 2e^x}} \end{array} \tag{40}$$

Example

Solve $y'' + y' = x + \sin 2x$.

$$\begin{array}{c} m^2 + m = 0 \implies m = \{0, -1\} \\ y_c = c_1 e^{-x} + c_2 e^0 \\ \underline{y_c = c_1 e^{-x} + c_2} \end{array}$$

The constant member (B) has to be avoided within y_p because it is used by y_c with arbitrary constant. If we use members with A, C and D, we are not able to cover x on the right side (there will be no member x on the left side once we substitute y_p into DE). That is the reason, why member Ex^2 has been introduced as well.

Member x from $(g(x))$ is x^k times member 1 from y_c. We have to include also x^2 into particular solution (see *case 2* from the lesson).

$$\begin{array}{rcl} y_p & = & Ax + B + C\sin 2x + D\cos 2x + Ex^2 \\ y_p' & = & A + 2C\cos 2x - 2D\sin 2x + 2Ex \\ y_p'' & = & -4C\sin 2x - 4D\cos 2x + 2E \end{array}$$

Substitute y_p' and y_p'' into DE:

$$\begin{aligned} (-4C\sin 2x - 4D\cos 2x + 2E)\ + & \\ +(A + 2C\cos 2x - 2D\sin 2x + 2Ex) &= x + \sin 2x \\ x(2E) + 1(A + 2E) + \cos 2x(-4D + 2C) + \sin 2x(-4C - 2D) &= x + \sin 2x \end{aligned} \qquad (41)$$

After solving $2E = 1 \implies E = 1/2$ and so on, the solution is written as

$$\underline{\underline{y = c_1 e^{-x} + c_2 + \frac{1}{2}x^2 - x - \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x}}.$$

Example

Solve $y'' - 3y' + 2y = xe^{-x}$.

Solving associated homogeneous DE gives

$$y_c = c_1 e^x + c_2 e^{2x}.$$

Building the particular solution from members of $g(x)$ and its derivativess:

$$\begin{aligned} y_p &= Axe^{-x} + Be^{-x} \\ y_p' &= A(1\cdot e^{-x} - xe^{-x}) - Be^{-x} = \\ &= Ae^{-x} - Axe^{-x} - Be^{-x} = \\ &= e^{-x}(A - B)xe^{-x}(-A) \\ y_p'' &= -Ae^{-x} - A(1\cdot e^{-x} - xe^{-x}) + Be^{-x} = \\ &= -Ae^{-x} - Ae^{-x} + Axe^{-x} + Be^{-x} = \\ &= e^{-x}(-2A + B) + xe^{-x}(A) \end{aligned}$$

Solving constant A and B for such values which satisfies DE:

$$\begin{aligned} e^{-x}(-2A + B) + xe^{-x}(A)\ - & \\ -3(e^{-x}(A - B) + xe^{-x}(-A))\ + & \\ +2(e^{-x}(B) + xe^{-x}(A)) &= xe^{-x} \end{aligned}$$

$$\begin{aligned} e^{-x}(-2A + B - 3A + 3B + 2B) + xe^{-x}(A + 3A + 2A) &= xe^{-x} \\ e^{-x}(-5A + 6B) + xe^{-x}(6A) &= xe^{-x} \end{aligned}$$

$$\begin{aligned} -5A + 6B &= 0 \\ 6A &= 1 \end{aligned}$$

After solving A and B we get the solution

$$\underline{\underline{y = c_1 e^x + c_2 e^{2x} + \frac{1}{6}xe^{-x} + \frac{5}{36}e^{-x}}}$$

Example

Solve $y'' + y' = x^2 + 2x$.

Solving associated homogeneous DE gives solution

$$y_c = c_1 + c_2 e^{-x}.$$

Let us prepare y_p, y_p' and y_p'':

$$\begin{aligned} y_p &= Ax^2 + Bx + C + Dx^3 \\ y_p' &= 2Ax + B + 3Dx^2 \\ y_p'' &= 2A + 6Dx \end{aligned}$$

Let us discuss those 3 above lines. The member C is already in y_c with arbitrary constant c_1, so it makes no sense to find value of B. If we feed y_p' and y_p'' into DE, we will observe that we can not pair member x^2 from the left to the right side. So we have to use also member x^3. Members x, x^2 are x^k times member of y_c.

$$\begin{aligned}(2A+6Dx)+(2Ax+B+3Dx^2) &= x^2+2x \\ 1(2A+B)+x(6D+2A)+x^2(3D) &= x^2+2x\end{aligned}$$

$$\left.\begin{aligned}2A+B&=0\\6D+2A&=2\\3D&=1\end{aligned}\right\} \implies \begin{aligned}A&=0\\B&=0\\D&=1/3\end{aligned}$$

And general solution is

$$\underline{\underline{y=c_1+c_2e^{-x}+\frac{1}{3}x^3.}}$$

Higher order linear differential equations

Undetermined coefficients—Annihilator approach

This is modified method of the method from the last lesson (Undetermined coefficients—superposition approach). The DE to be solved has again the same limitations (constant coefficients and restrictions on the right side). Annihilator approach finds y_c and y_p by means of operators explained further. Finally the values of arbitrary constants of particular solution have to be found as was explained.

Prior to explain the method itself we need to introduce some new terms we will use later.

Differential and Linear differential operator

An operator is a mathematical device which converts one function into another. For example the operator $'$ (differential operator) converts $f(x)$ into a new function $f'(x)$. $\int f(t)\,dt$ converts $f(t)$ into new function $F(x)$.

We also use letter D to denote the operation of differentiation. D is called **differential operator.**

$$\begin{aligned} D^0 y &= y \\ D^1 y &= y' \\ D^2 y &= y'' \\ &\vdots \\ D^n y &= y^{(n)} \end{aligned}$$

Linear differential operator

If we use differential operator D we may form a linear combination of differential operators of orders 0 to n:

$$a_0 + a_1 D + a_2 D + \cdots + a_n D^n$$

Thus we a have a handy tool which helps us also to generalize some rules into sample manner. For example if we work with operator in above polynomial form, we may rely also on polynomial behaviour, e.g.:

$$(D+2)(D+3) = (D+3)(D+2) = (D^2+5D+6),$$
$$(D^2+3)(x^3+\cos x) = D^2(x^3+\cos x) + 3(x^3+\cos x)$$

Annihilator operator

If L is linear differential operator such that

$$L(f(x)) = 0$$

then L is said to be annihilator. For example

- if $y = k$ then D is annihilator ($D(k) = 0$), k is a constant,
- if $y = x$ then D^2 is annihilator ($D^2(x) = 0$),
- if $y = x^{n-1}$ then D^n is annihilator.

D^n annihilates not only x^{n-1}, but all members of polygon

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

A function $e^{\alpha x}$ is annihilated by $(D-\alpha)$:

$$(D-\alpha)e^{\alpha x} = De^{\alpha x} - \alpha e^{\alpha x} = \alpha e^{\alpha x} - \alpha e^{\alpha x} = 0$$

$(D-\alpha)^n$ annihilates each of the member

$$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \cdots, x^{n-1} e^{\alpha x}$$

$(D^2+\beta^2)$ annihilates $\cos\beta x$ and also $\sin\beta x$:

$$(D^2+\beta^2)\cos\beta x = 0, \quad (D^2+\beta^2)\sin\beta x = 0.$$

We are often interested in **annihilating a sum of two or more functions** y_1, y_2, ... It can be shown that

- if $L(y_1)=0$ and $L(y_2)=0$ then L annihilates also linear combination $c_1y_1+c_2y_2$.
- if $L_1(y_1)=0$ and $L_2(y_2)=0$ then L_1L_2 annihilates sum $c_1y_1+c_2y_2$.

Example:

$7-x$ is being annihilated by D^2
$\sin 4x$ is being annihilated by D^2+16
Then $D^2(D^2+16)$ annihilates the linear combination $7-x+6\sin 4x$

The differential operator which annihilates given function is not unique. For example $D^2(x)=0$. But also $D^3(x)=0$. **We want the operator of the lowest possible order.**

Solving nonhomogeneous DE

Homogeneous high order DE can be written also as $L(y)=0$ and nonhomogeneous as $L(y)=g(x)$ where L is a proper differential operator.

Example:

$$\begin{aligned} y''+3y'+2y &= 0 \quad \text{can be written as} \\ (D^2+3D+2)y &= 0 \end{aligned}$$

The right side containing $g(x)$ can be annihilated by L_1:

$$\begin{aligned} L(y) &= g(x) \\ L_1L(y) &= L_1(g_x)) \quad \text{right side is being annihilated} \\ L_1L(y) &= 0 \end{aligned}$$

If we solve $L_1L(y)=0$ we get an instance of solution $y=y_c+y_p$.

Then we have to distinguish terms which belong to particular solution y_p and find constants for all these terms.

Example

Solve $y''+4y'+4y=2x+6$

Step 1

This step is voluntary and rather serves to bring more light into the method. We will find y_c as we are used to:

$$m^2+4m+4=0 \implies m=\{-2,-2\}$$
$$\underline{y_c=c_1e^{-2x}+c_2xe^{-2x}}$$

Step 2

Apply annihilators against $g(x)$.

$$\begin{aligned} (D^2+4D+4)y &= 2x+6 \\ D^2(D^2+4D+4)y &= D^2(2x+6) \\ D^2(D^2+4D+4)y &= 0 \\ m^2(m^2+4m+4) &= 0 \implies m=\{0,0\},\{-2,-2\} \end{aligned}$$

It can be seen that the solution $m=\{-2,-2\}$ belongs to complementary function y_c and $m=\{0,0\}$ belongs to particular solution y_p.

$$y=c_1e^{-2x}+c_2xe^{-2x}+c_3e^0+c_4e^0x$$
$$\underline{y=c_1e^{-2x}+c_2xe^{-2x}+c_3+c_4x}$$

Step 3

The particular solution is not supposed to have its members multiplied by arbitrary constants. We have to find values c_3 and c_4 in such way, that the solution satisfies DE. To do so, we will use method of undeterminated coefficients as in previous lesson. We know that y_p is a solution of DE. So we can feed $y_p=A+Bx$ and its derivatives into DE and find constants A, B: $A=1$, $B=\frac{1}{2}$.

$$y = 1 + \frac{1}{2}x + c_1 e^{2x} + c_2 e^{-2x} x$$

Example

Solve $y''' - y'' + y' - y = xe^x - e^{-x} + 7$

Step 1

$$m^3 - m^2 + m - 1 = 0 \quad \Longrightarrow m = \{1, i, -i\}$$
$$y_c = c_1 e^x + c_2 \cos x + c_3 \sin x$$

Note that the imaginary roots come in conjugate pairs.

Step 2

Apply annihilators against $g(x)$.

$$\begin{array}{rl} 7 & \dots D \\ e^{-x} & \dots (D+1) \\ xe^x & \dots (D-1)^2 \end{array}$$

$$\begin{aligned} (D^3 - D^2 + D - 1)y &= xe^x - e^{-x} + 7 \\ D(D+1)(D-1)^2(D^3 - D^2 + D - 1)y &= D(D+1)(D-1)^2(xe^x - e^{-x} + 7) \\ D(D+1)(D-1)^2(D^3 - D^2 + D - 1)y &= 0 \\ m(m+1)(m-1)^2(m^3 - m^2 + m - 1) &= 0 \Longrightarrow m = \{1, i, -i\}, \{0, -1, 1, 1\} \end{aligned}$$

$$y_p(x) = Ae^0 + Be^{-x} + Ce^x x + De^x x^2$$

In step 1 the members of complementary function y_c are found from auxiliary equation. However even if step 1 is skipped, it should be obvious which roots belong to y_c and which roots belong to y_p from step 2 itself.

After expressing y_p' and y_p'' we can feed them into DE and find constants A, B, C and D of particular solution. Finally we can form

$$y_c = c_1 e^x + c_2 \cos x + c_3 \sin x - 7 + \frac{1}{4}e^{-x} - \frac{1}{2}xe^x + \frac{1}{4}x^2 e^x.$$

Example

Solve $y'' - y' + y = x^2$

First we rewrite the DE by means of differential operator D and then we have to ask, what is annihilator for x^2 on the right side? Is it D? It is not: D annihilates only a constant. We have to use D^3 to annihilate x^2.

$$\begin{gathered} D^2 y + Dy + y = x^2 \\ D^3(D^2 y + Dy + y) = D^3(x^2) \\ D^3(D^2 y + Dy + y) = 0 \\ m^3(m^2 + m + 1) = 0 \end{gathered}$$

The member m^3 belongs to the particular solution y_p and roots from $m^2 + m + 1$ will form complementary function y_c. To find roots we might use calculator able to solve quadratic equation or we might use quadratic formula being taught at high school.

The found roots are $m = \{0,\ 0,\ 0,\ -1/2 + i\sqrt{3}/2,\ -1/2 - i\sqrt{3}/2\}$. The general solution can be formed as

$$y = y_c + y_p = c_1 e^{-1/2 - i\sqrt{3}/2} + c_2 e^{-1/2 + i\sqrt{3}/2} + c_3 + c_4 x + c_5 x^2$$

The first members involve imaginary numbers and might be also rewritten by means of $\sin()$ and $\cos()$ to avoid complex numbers. These roots comes in conjugate pairs $\alpha + i\beta$ and $\alpha - i\beta$, so they do not repeat. The next three members would repeat based on the value of the root $m = 0$, so they are multiplied by x and x^2.

The job is not done yet, since we have to find values of constants c_3, c_4, c_5 which are part of particular solution. Note that we have 2nd order DE, so we expect to have two arbitrary constants, not five.

These constants can be obtained by forming particular solution in a more convenient way $y_p = A + Bx + Cx^2$, preparing $y_p',\ y_p''$ ans substituting into DE. It will be found that $A = 0,\ B = -2,\ C = 1$.

$$\underline{\underline{y = c_1 e^{-1/2 - i\sqrt{3}/2} + c_2 e^{-1/2 + i\sqrt{3}/2} - 2x + x^2}}$$

Higher order linear differential equations

Variation of parameters

The previous methods (method of undetermined coefficients to find particular solution y_p) have two weaknesses:

- the kind of members on the right side ($g(x)$) is restricted
- the DE must have *constant* coefficients.

Note: regarding the first issue, in some cases of DE with non-constant coefficients, we are still able to use method of undetermined coefficients.

Assuming that we are able to find solution $y_c(x)$ of associated homogeneous DE (i.e. to find complementary function), this method is able **to find particular solution y_p of linear DE with non-constant coefficients with no restrictions on the right side.**

The method will be shown below for differential equation of 2nd order but can be used for higher order DE.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

We assume that there are no troubles with finding the two linearly independent solutions $y_1(x)$ and $y_2(x)$ of DE. With them we form equation for $y_p(x)$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{42}$$

The above particular solution is expressed in terms of solutions $y_1(x)$ and $y_2(x)$. Functions $u_1(x)$ and $u_2(x)$ are *unknown* and have to be determined.

The derivatives of $y_p(x)$ are

$$\begin{aligned} y_p'(x) &= u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2 = \\ &= (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2) \\ y_p''(x) &= (u_1y_1'' + u_2y_2'') + (u_1'y_1' + u_2'y_2') + (u_1'y_1 + u_2'y_2)' \end{aligned}$$

Substituting solution $y_p(x)$ and its derivatives $y_p'(x)$ and $y_p''(x)$ into DE

$$\begin{aligned} a_2(u_1y_1'' + u_2y_2'') + a_2(u_1'y_1' + u_2'y_2') + a_2(u_1'y_1 + u_2'y_2)' \quad &+ \\ +a_1(u_1y_1' + u_2y_2') + a_1(u_1'y_1 + u_2'y_2) + a_0(u_1y_1 + u_2y_2) \quad &= g(x) \end{aligned}$$

This equation can be written as

$$\begin{aligned} u_1(a_2y_1'' + a_1y_1' + a_0y_1) + u_2(a_2y_2'' + a_1y_2' + a_0y_2) \quad &+ \\ +a_2(u_1'y_1' + u_2'y_2') + a_2(u_1'y_1 + u_2'y_2)' + a_1(u_1'y_1 + u_2'y_2) \quad &= g(x) \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are solutions of homogeneous DE, the first two parantheses equal zero. Then, if we request that

$$u_1'y_1 + u_2'y_2 = 0, \tag{43}$$

the remaining members have to hold equality

$$u_1'y_1' + u_2'y_2' = \frac{g(x)}{a_2}$$

or

$$u_1'y_1' + u_2'y_2' = f(x) \tag{44}$$

for DE in the *standard form* of

$$y'' + P(x)y' + Q(x)y = f(x). \tag{45}$$

The pair of equations (43) and (44) can be solved for u_1' and u_2' using common algebraic methods. The other solution is by Cramer's law:

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W}, \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}, \tag{46}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}. \tag{47}$$

The above determinant W is Wronskian and W_1 and W_2 are derived from W using Cramer's law. In our case we have two equations (43) and (44) and determinant with two columns for 2nd order DE. But the method can be generalized to linear nth order differential equations.

Example

Solve DE $ty'' - (t+1)y' + y = t^2$ with provided solutions $y_1(t) = e^t$ and $y_2(t) = t + 1$.

The DE does not have constant coefficients. Such DE is not easily solved and we are being provided solutions of associated homogeneous DE. The particular solution $y_p(x)$ will be computed by means of method *variation of parameters* from $y_c(x)$.

First let us divide DE by t to bring DE into standard form. Because the method expects that we work with DE in standard form of $y'' + P(t)y' + Q(t)y = f(t)$.

$$\begin{aligned} y'' + \frac{t+1}{t} y' + \frac{1}{t} y &= t \\ y_1(t) = e^t \qquad y_1'(t) &= e^t \\ y_2(t) = t+1 \qquad y_2'(t) &= 1 \end{aligned}$$

Now we have to form Wronskian W and W_1, W_2 as the method expects:

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix} = e^t - e^t(t+1) = -e^t \cdot t \\ u_1'(t) &= \frac{W_1}{W} = -\frac{y_2 f(t)}{W} = -\frac{(t+1)t}{-e^t \cdot t} = \frac{t+1}{e^t} \\ u_1(t) &= (-t-1)e^{-t} - e^{-t} \\ u_2'(t) &= \frac{W_2}{W} = \frac{y_1 f(t)}{W} = \frac{e^t \cdot t}{W} = \frac{e^t \cdot t}{-e^t \cdot t} = -1 \\ u_2(t) &= -t \\ y_p(t) &= u_1 y_1 + u_2 y_2 = [(-t-1)e^{-t} - e^{-t}]e^t + (-t)(t+1) = -t^2 - 2t - 2 \end{aligned}$$

We were given $y_c(t)$, we have found $y_p(t)$, so we can collect the general solution:

$$\underline{\underline{y(t) = c_1 e^t + c_2(t+1) - t^2 - 2t - 2}}$$

Example

Find the general solution of $y'' + y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The DE can not be solved by the method of undetermined coefficients, because on the right side is a member with infinite number of linearly independent derivatives.

Solving associated homogeneous DE brings complementary function

$$y_c = c_1 \cos x + c_2 \sin x.$$

That means for the purpose of $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ we assign the two solution as

$$\begin{aligned} y_1 &= \cos x, \\ y_2 &= \sin x. \end{aligned}$$

Using the equation (46) is a convenient way to get u_1 and u_2:

$$\begin{aligned} u_1'(x) &= -\frac{y_2 f(x)}{W} = -\frac{\sin x \tan x}{\cos x \cos x - \sin x(-\sin x)} = -\frac{\sin x \tan x}{\cos^2 x + \sin^2 x} = \frac{-\sin x \tan x}{1} = \\ &= -\frac{\sin^2 x}{\cos x} \end{aligned}$$

$$u_2'(x) = \frac{y_1 f(x)}{W} = \frac{\cos x \tan x}{1} = \sin x$$

After integrating u_1' and u_2':

$$u_1(x) = -\log(\sec x + \tan x) + \sin x, \quad u_2(x) = -\cos x.$$

Since we have $u_1(x)$ and $u_2(x)$, we are able to build $y_p(x)$ from (42):

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\ y_p(x) &= (-\log(\sec x + \tan x) + \sin x)\cos x + (-\cos x)\sin x \\ y_p(x) &= -\log(\sec x + \tan x)\cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \end{aligned}$$

Combining y_c with y_p we obtain the general solution of DE:

$$\underline{\underline{y = c_1 \cos x + c_2 \sin x - \log(\sec x + \tan x)\cos x}}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Example

Homogeneous differential equations $x^2y'' + xy' - y = 0$ has two solutions $y_1 = x$ and $y_2 = x^{-1}$. Find the general solution of $x^2y'' + xy' - y = x,\ x \neq 0$.

Using the equation (46) is a convenient way to get u_1 and u_2:

$$\begin{aligned} u_1' &= -\frac{y_2 f(x)}{W} = -\frac{x^{-1}x/x^2}{W} = -\frac{x^{-2}}{(x(-x^{-2}) - (x^{-1}\cdot 1)} = -\frac{x^{-2}}{-2x^{-1}} = \\ &= -\frac{1}{2x} \end{aligned}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{x \cdot x/x^2}{-2x^{-1}} = \frac{1}{-2x^{-1}} = -\frac{1}{2}x$$

After integrating u_1' and u_2':

$$u_1(x) = \frac{1}{2}\log x, \quad u_2(x) = -\frac{x^2}{4}.$$

Since we have $u_1(x)$ and $u_2(x)$ we are able to build $y_p(x)$ from (42):

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\ y_p(x) &= \frac{x}{2}\log x - \frac{x}{4} \end{aligned}$$

Combining y_c with y_p we obtain the general solution of DE:

$$\underline{\underline{y = c_1 x + c_2 x^{-1} + \frac{x}{2}\log x - \frac{x}{4}}}.$$

Higher order linear differential equations

Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=0$$
$$4x^2y''+8xy'+y=0$$

are examples of Cauchy-Euler differential equations. A linear differential equation of the form

$$a_nx^n\frac{d^ny}{dx^n}+a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1x\frac{dy}{dx}+a_0y=g(x),$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known as a Cauchy-Euler equation. The important observation is that coefficient x^k matches the order of differentiation.

Cauchy-Euler differential equations have solutions in format $y(x)=x^m$ and are easily solved. The solution is quite straight-forward. We lay expected solution $y(x)=x^m$ and feed the solution with its derivatives $y'(x)$ and $y''(x)$ into DE. Solving leads to separating x^m and **auxiliary equation (see the examples below).** The roots m have to be determined from auxiliary equation.

Once we have roots m, we have to deal with 3 possibilities:

The roots are distinct, do not repeat

Solution is in the form $y(x)=c_1x^{m_1}+c_2x^{m_2}$.

Roots repeat

E.g. $m_1=2$, $m_2=2$. The solutions can not be linearly dependent. *The Reduction of order method* determines from the first solution $y_1(x)=x^m$ the second solution as $y_2(x)=x^m\log x$.

Roots are complex numbers

It is an analogy to the above. The imaginary roots come in conjugate pairs and solution can be written as $y(x)=c_1x^{\alpha-i\beta}+c_2x^{\alpha+i\beta}$. However we often prefer to work with expression without complex numbers. The other form is $y=C_1x^\alpha\cos(\beta\log x)+C_2x^\alpha\sin(\beta\log x)$.

Note: although the text describes mainly second order differential equation, all the ideas can be easily extended to analogous differential equation of any order.

Example

Solve differential equation $x^2y''-3xy'+4y=0$ with initial values $y(1)=1$ and $y'(1)=2$.

We assume that the solution is in the form $y(x)=x^m$. Then

$$\begin{aligned} y(x) &= x^m, \\ y'(x) &= mx^{m-1}, \\ y''(x) &= m(m-1)x^{m-2}. \end{aligned}$$

Let us use the solution within given differential equation:

$$\begin{aligned} x^2\cdot m(m-1)x^{m-2}-3x\cdot mx^{m-1}+4\cdot x^m &= 0 \\ x^m((m^2-m)-3m+4) &= 0 \\ x^m(m^2-4m+4) &= 0 \\ m &= \{2,2\} \end{aligned}$$

The solution of differential equation is

$$\underline{y_c(x)=c_1x^2+c_2x^2\log x.} \tag{48}$$

In order to find values of c_1 and c_2 according to initial values, we need also y_c':

$$\begin{aligned} y_c'(x) &= 2c_1 x + 2c_2 x \log x + c_2 x^2 \frac{1}{x} = \\ &= 2c_1 x + 2c_2 x \log x + c_2 x. \end{aligned} \tag{49}$$

For $y(1) = 1$ and $y'(1) = 2$ we get from (48) and (49) $c_1 = 1$, $c_2 = 0$. The solution is then

$$\underline{\underline{y(x) = x^2}}.$$

Example

Solve differential equation $x^2 y'' - 2xy' + 2y = x^3, \quad x > 0$

Because the differential equation is in the form of Cauchy-Euler DE we assume that the solution is in the form of $y(x) = x^m$. Then

$$\begin{aligned} y(x) &= x^m, \\ y'(x) &= mx^{m-1}, \\ y''(x) &= m(m-1)x^{m-2}. \end{aligned}$$

Let us use the solution within associated homogeneous differential equation to obtain complementary function $y_c(x)$:

$$\begin{aligned} x^2 \cdot m(m-1)x^{m-2} - 2x \cdot mx^{m-1} + 2 \cdot x^m &= 0 \\ x^m(m(m-1) - 2m + 2) &= 0 \\ x^m(m^2 - 3m + 2) &= 0 \\ x^m(m-2)(m-1) &= 0 \\ m &= \{1, 2\} \end{aligned}$$

The complementary function $y_c(x)$ (i.e. the solution of associated homogeneous differential equation) is

$$\underline{y_c(x) = c_1 x + c_2 x^2}. \tag{50}$$

We have y_c but we have no y_p, so the solution is still incomplete. The differential equation has no constant coefficients, so we can not use *method of undeterminated coefficients*. We have to employ *method of variation of parameters* to determine particular solution y_p.

Let us take $y_1(x)$ and $y_2(x)$ from $y_c(x)$ and form particular solution $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$:

$$\begin{aligned} y_1(x) = x, &\quad y_2(x) = x^2 \\ y_1'(x) = 1, &\quad y_2'(x) = 2x \end{aligned}$$

We have to find $u_1'(x)$ and $u_2'(x)$ as method of variation of parameters requires ($f(x) = x^3/x^2 = x$ for differential equation in standard form):

$$\begin{aligned} u_1'(x) &= \frac{W_1}{W} = -\frac{x^2 x}{2x^2 - x^2} = -\frac{x^3}{x^2} = -x, \\ u_2'(x) &= \frac{W_2}{W} = \frac{x \cdot x}{x^2} = 1 \end{aligned}$$

After integrating $u_1'(x)$ and $u_2'(x)$ we may form particular and general solution:

$$\underline{y_p(x) = -\frac{x^2}{2} x + x \cdot x^2 = \frac{1}{2} x^3}$$

$$\underline{\underline{y(x) = c_1 x + c_2 x^2 + \frac{1}{2} x^3}}.$$

Example

Solve differential equation $x^2y'' + 4xy' - 4y = 3$

First, let us solve associated homogeneous DE $x^2y'' + 4xy' - 4y = 0$ for y_c, then particular solution y_p.

We expect the solution in a form of $y = x^m$, then $y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$.

$$\begin{aligned} x^2\, m(m-1)x^{m-2} + 4x\, mx^{m-1} - 4\, x^m = 0 \\ x^m[m(m-1) + 4m - 4] = 0 \\ x^m(m^2 + 3m - 4) = 0 \implies m = \{-4, 1\} \end{aligned}$$

The complementary function is $y_c = c_1x + c_2x^{-4}$ and particular solution $y_p = A$ has to be determined and it is pretty obvious that $A = -3/4$ (test yourself that $y_p = -3/4$ satisfies given DE).

$$\underline{\underline{y = c_1x + c_2x^{-4} - \frac{3}{4}}}$$

Higher order differential equations

Solving system of linear DE with constant coefficients

Examples of systems of linear differential equations:

$$\left.\begin{aligned} 2x' + 3x + 5y' - y &= e^t \\ x' - x + 3y' + y &= \sin t \end{aligned}\right\} \quad \text{(a)}$$

$$\left.\begin{aligned} (D^2 + 3D - 1)x + y &= 6 + t^2 \\ (D + 2)x - (D^2 + D)y &= t \end{aligned}\right\} \quad \text{(b)}$$

Systems of linear DE with constant coefficients leads readily to solutions by means of operator D.

The above system (a) can be rewritten into

$$\begin{aligned} (2D + 3)x + (5D - 1)y &= e^t, \\ (D - 1)x + (3D + 1)y &= \sin t. \end{aligned}$$

The pair of equations

$$\begin{aligned} f_1(D)x + g_1(D)x &= h_1(t), \\ f_2(D)x + g_2(D)x &= h_2(t), \end{aligned}$$

where D is operator d/dt, and coefficients in front of x and y are linear differential operators, is called a system of two linear DE.

We have learnt already that **linear differential operator obey all the rules of algebra. So we will use the rules and methods which are used in solving an algebraic system of equations**, e. g.:

$$\begin{aligned} 2x + 3y &= 7, \\ 3x - 2y &= 4. \end{aligned}$$

The above system can be solved by elimination and we will use such approach also to systems of DE, which are written using the operator D.

Example

Solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + 2y \end{aligned}$$

First, we rewrite the above equations by means of operator D:

$$Dx - y = 0 \tag{51}$$
$$Dy + x - 2y = 0 \tag{52}$$

$$\begin{aligned} Dx - y &= 0 \qquad / \cdot (D - 2) \\ y(D - 2) + x &= 0 \end{aligned}$$

$$\begin{aligned} D(D - 2)x - (D - 2)y &= 0 \\ x + (D - 2)y &= 0 \end{aligned}$$

Let us add both equations to eliminate $(D - 2)y$ and we get

$$
\begin{aligned}
(D^2-2D)x+x&=0\\
D^2x-2Dx+x&=0\\
x''-2x'+x&=0\\
m^2-2m+1&=0 \implies m=\{1,1\}
\end{aligned}
$$

So far, we are able to express x:

$$\underline{\underline{x=c_1e^t+c_2te^t}}$$

Now we may substitute x into (51) to get y:

$$
\begin{aligned}
&D(c_1e^t+c_2te^t)-y=0\\
&\underline{\underline{y=c_1e^t+c_2te^t+c_2e^t}}
\end{aligned}
$$

If we substitute x, y, y' into (52) we will find that the equality holds, so above determined x and y are complete solution.

Example

Solve the system of differential equations

$$3\frac{dx}{dt}+3x+2y=e^t \tag{53}$$

$$4x-3\frac{dy}{dt}+3y=3t \tag{54}$$

First, we rewrite the above equations by means of operator D:

$$
\begin{aligned}
3Dx+3x+2y&=e^t\\
4x-3Dy+3y&=3t
\end{aligned}
$$

$$
\begin{aligned}
x(3D+3)+2y&=e^t \qquad /\cdot(3D-3)\\
4x-y(3D-3)&=3t \qquad /\cdot 2
\end{aligned}
$$

$$
\begin{aligned}
x(3D+3)(3D-3)+2y(3D-3)&=e^t(3D-3)\\
8x-2y(3D-3)&=6t
\end{aligned}
$$

Let us add both last equations to eliminate $2y(3D-3)$:

$$
\begin{aligned}
x(3D+3)(3D-3)+8x&=6t\\
x(9D^2-9)+8x&=6t\\
9x''-9x+8x&=6t\\
9x''-x&=6t\\
9m^2-1=0 &\implies m=\{\frac{1}{3},-\frac{1}{3}\}
\end{aligned}
$$

So far, we are able to collect x:

$$\underline{\underline{x=c_1e^{1/3t}+c_2e^{-1/3t}-6t}}$$

Substituting x' into (53) brings y:

$$\underline{\underline{y=-2c_1e^{1/3t}-c_2e^{-1/3t}+9+\frac{1}{2}e^t+9t}}$$

If we substitute x, y, y' into (54) we will observe that the equality holds, so above determined x and y are complete solution.

Example

Solve the system of differential equations

$$2\,\frac{dx}{dt}+\frac{dy}{dt}-x=e^t \tag{55}$$

$$3\,\frac{dx}{dt}+2\,\frac{dy}{dt}+y=t \tag{56}$$

First, we rewrite the above equations by means of operator D:

$$\begin{aligned} 2Dx+Dy-x&=e^t\\ 3Dx+2Dy+y&=t \end{aligned}$$

$$\begin{aligned} 2Dx-x+Dy&=e^t \qquad /\cdot(2D+1)\\ 3Dx+(2D+1)y&=t \qquad /\cdot(-D) \end{aligned}$$

$$\begin{aligned} 2D(2D+1)x-(2D+1)x+D(2D+1)y&=e^t(2D+1)\\ -3D^2x-D(2D+1)y&=t(-D) \end{aligned}$$

Let us add both equations to eliminate $D(2D+1)y$.

$$\begin{aligned} (4D^2+2D)x-(2D+1)x-3D^2x&=e^t(2D+1)-1\\ 4D^2x+2Dx-2Dx-x-3D^2x&=2De^t+e^t-1\\ D^2x-x&=2e^t+e^t-1\\ x''-x&=3e^t-1 \end{aligned}$$

The solution for x is

$$\underline{\underline{x(t)=c_1e^t+c_2e^{-t}+\frac{3}{2}\,e^tt+1}}.$$

Let us solve $x''-x=3e^t-1$ by annihilator approach:

$$\begin{aligned} (D^2-1)x&=3e^t-1\\ (D^2-1)x(D-1)D&=(3e^t-1)(D-1)D\\ (D^2-1)x(D-1)D&=0\\ (m^2-1)(m-1)m&=0 \implies m=\{1,-1,1,0\} \end{aligned}$$

The first two roots belong to complementary function the next two are to form particular solution. So we rewrite constants $c_3,\ c_4$ to more convenient $A,\ B$ and solve them.

$$\begin{aligned} x(t)&=c_1e^t+c_2e^{-t}+c_3e^tt+c_4e^0\\ x_p(t)&=Ae^tt+B \end{aligned}$$

The constants A, B of particular solution have to be found by means of method of undetermined coefficients. For that purpose we need $x_p'(t),\ x_p''(t)$

$$\begin{aligned} x_p'(t)&=A(e^t\cdot 1+e^tt)=Ae^t+Ae^tt\\ x_p''(t)&=Ae^t+Ae^tt+Ae^t=2Ae^t+Ae^tt \end{aligned}$$

Now we can substitute $x_p(t),\ x_p''(t)$ into DE $x''-x=3e^t-1$ to determine $A,\ B$.

$$\begin{aligned} 2Ae^t+Ae^tt-Ae^tt-B&=3e^t-1\\ e^t(2A)-B&=3e^t-1 \implies 2A=3,\ B=1 \end{aligned}$$

Finaly we may collect $x(t)$:

$$\underline{x(t)=c_1e^t+c_2e^{-t}+\frac{3}{2}\,e^tt+1}.$$

To determine $y(t)$ we can use (55). For that purpose we have to differentiate $x(t)$ to get $x'(t)$, then from (55) express $y'(t)$ and integrate $y'(t)$.

$$x'(t) = c_1 e^t - c_2 e^{-t} + \frac{3}{2}(e^t t + e^t)$$
$$y'(t) = e^t(-c_1 - 2) + e^{-t}(3c_2) - e^t t(\frac{3}{2}) + 1$$
$$y(t) = e^t(-c_1 - \frac{1}{2}) - e^{-t}(3c_2) - \frac{3}{2} e^t t + t + d_1$$

The solutions $x(t)$, $y(t)$ have to comply both equations (55), (56). The constant d_1 can be determined again by method of undetermined coefficients from (56).

$$\underline{\underline{y(t) = e^t(-c_1 - \frac{1}{2}) - e^{-t}(3c_2) - \frac{3}{2} e^t t + t - 2}}$$

If we substitute x, x', y, y' into system of DE, it will confirm that the equalities hold.

Example

Solve the system of differential equations

$$\frac{dx}{dt} = 3e^{-t}$$
$$\frac{dy}{dt} = x + y$$

First, rewrite the above equations by means of operator D and then we solve as an basic algebra task (to eliminate x).

$$Dx = 3e^{-t}$$
$$Dy = x + y$$

$$Dx = 3e^{-t}$$
$$Dy - x - y = 0 \qquad / \cdot D \tag{57}$$

$$Dx = 3e^{-t}$$
$$D(Dy - x - y) = 0$$

$$Dx = 3e^{-t} \tag{58}$$
$$D^2 y - Dx - Dy = 0 \tag{59}$$

$$D^2 y - Dy = 3e^{-t} \qquad \text{Note: (58) + (59)}$$

Finally we have to solve $y'' - y' = 3e^{-t}$ which is solved by means of auxiliary equation $(m^2 - m) = 0 \implies m = \{0, 1\}$, so we can express y_c

$$y_c = c_1 + c_2 e^t$$

and y_p has to be determined. That can be accomplished by the method of undetermined coefficients, because considering the members on the right side (e^{-t}) we can expect particular solution containing e^{-t} and its derivatives:

$$\left.\begin{aligned} y_p &= Ae^{-t} \\ y_p' &= -Ae^{-t} \\ y_p'' &= Ae^{-t} \end{aligned}\right\} \qquad Ae^{-t} + Ae^{-t} = 3e^{-t} \implies A = \frac{3}{2}$$

Then

$$\underline{\underline{y = c_1 + c_2 e^t + \frac{3}{2} e^{-t}.}}$$

Now there are more ways how to find x. We can use (57). For that purpose we have to differentiate y.

$$
\begin{aligned}
&Dy = c_2e^t - \frac{3}{2}e^{-t} \\
&c_2e^t - \frac{3}{2}e^{-t} - x - (c_1 + c_2e^t + \frac{3}{2}e^{-t}) = 0 \\
&e^t(c_2 - c_2) + e^{-t}(-\frac{3}{2} - \frac{3}{2}) - c_1 = x
\end{aligned}
$$

When simplified we have x

$$\underline{\underline{x = -c_1 - 3e^{-t}}}$$

Since we are solving system of DE, it is good idea to test the solutions against both given DE to ensure that no part of solution has been lost.

Higher order differential equations

Nonlinear DE of higher order

Examples:

$$xy'' + 2y' + x = 1 \tag{60}$$
$$y'' + (y')^3 y = 0 \tag{61}$$

When we are describing real world problems, the linear DE might be not sufficient to involve required dependencies, so there is a need for nonlinear DE.

The nonlinear DE of higher order are difficult to solve and often can not be solved analytically.

Regarding *linear* DE, it was told that if we know solution y_1 and y_2 the linear combination is also a solution. Such statement may not hold with *nonlinear* DE. For example differential equation $(y'')^2 = y^2$ has a solution $y_1 = e^x$ and $y_2 = \cos x$. You can test that linear combination is not a solution.

Nonlinear DE can not be solved by reduction of order method (from solution y_1); consider that we are even not able to bring non-linear DE into standard form of linear DE in order to locate $P(x)$. But nonlinear DE can be solved numerically and one of the method will be shown below.

Substitution to reduct the order

In *some* cases the solution of nonlinear higher order DE can be made by **reduction of order by means of substitution:** in both above examples (60) and (61), substitution $u = y'$ brings DE into the first order DE which is solved easily.

Let us distinguish between two cases:

1. The independent **variable y is missing:** $F(x,\ y',\ y'') = 0$; example (60) above.
2. The dependent **variable x is missing:** $F(y,\ y',\ y'') = 0$; example (61) above.

The **first case** is simple. We lay a substitution

$$u = y', \quad u' = \frac{du}{dx}$$

and the second order DE is converted into the first order DE. Solve the first order linear DE (instead of y we deal with u).

If we try to use the same approach for the **second case,** we will fail with solving. Because $u = y'$, considering suggested $u' = du/dx$ and at the same time in the DE is only y, u and u' left, it would be more complicated than before. We have to employ chain rule to get rid of dx:

$$u' = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}y' = \frac{du}{dy}u.$$

Then dx is eliminated from DE and DE is solved as first order DE of u in terms of y.

Example

Solve $xy'' + 2y' + x = 1$ as a nonlinear DE with initial values of $y(1) = 2$ and $y'(1) = 1$.

Although it is linear DE with *non-constant* coefficients and can be solved also as a Cauchy-Euler DE, we will demonstrate substitution as it was nonlinear DE. Dependent variable is missing. We can use substitution $u = y'$ to reduce the order.

$$xu' + 2u + x = 1, \qquad u = y'$$

Let us solve for u:

$$
\begin{aligned}
u' + \frac{2}{x}u + 1 &= \frac{1}{x} \\
u' + \frac{2}{x}u &= \frac{1}{x} - 1 \quad \text{or} \\
\frac{du}{dx} + \frac{2}{x}u &= \frac{1}{x} - 1
\end{aligned}
$$

It is a linear DE of first order (we might rather rewrite it with y instead of u to avoid confusion prior solving), which has a solution

Solution of the above first order DE:

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x} - 1$$

It is first order DE which is linear. Now it is in standard form, so we can take $P(x) = 2/x$ and built integrating factor $\mu(x) = e^{P(x)\,dx}$ and multiply both sides by integrating factor.

$$
\begin{aligned}
\mu(x) &= e^{P(x)\,dx} = e^{2\int \frac{1}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2 \\
x^2\frac{dy}{dx} + x^2\frac{2}{x}y &= x^2(\frac{1}{x} - 1)
\end{aligned}
$$

We know that members on the left side are product of $\frac{d}{dx}\mu(x)\cdot y$ and that can be also tested (product rule). Then integrate both sides:

$$
\begin{aligned}
(x^2)(y) &= \int (x - x^2)\,dx \\
x^2 y &= \frac{x^2}{2} - \frac{x^3}{3} + c_1 \\
y &= \frac{1}{2} - \frac{x}{3} + c_1\frac{1}{x^2}
\end{aligned}
$$

$$u(x) = c_1\frac{1}{x^2} - \frac{x}{3} + \frac{1}{2}$$

We were given initial value $y'(1) = 1 \implies u(1) = 1$

$$1 = c_1\frac{1}{1} - \frac{1}{3} + \frac{1}{2} \implies c_1 = \frac{5}{6}$$

So we have

$$u(x) = \frac{5}{6}\frac{1}{x^2} - \frac{x}{3} + \frac{1}{2}.$$

Let us integrate $u(x)$ to get $y(x)$:

$$
\begin{aligned}
y(x) = \int u(x)\,dx &= \int (\frac{5}{6}\frac{1}{x^2} - \frac{x}{3} + \frac{1}{2})dx \\
&= -\frac{5}{6}x^{-1} - \frac{1}{6}x^2 + \frac{1}{2}x + c_2
\end{aligned}
$$

Let us use initial value $y(1) = 2$ to get c_2:

$$2 = -\frac{5}{6}1^{-1} - \frac{1}{6}1^2 + \frac{1}{2}1 + c_2 \implies c_2 = \frac{5}{2}$$

The solution is

$$y(x) = -\frac{5}{6}\frac{1}{x} - \frac{1}{6}x^2 + \frac{1}{2}x + \frac{5}{2}.$$

Example

Solve nonlinear DE $xy'' = y' + x(y')^2$.

Let us use substitution $u = y'$. Then

$$\begin{aligned} xu' &= u + xu^2 \\ u' - \frac{1}{x}\,u &= u^2 \\ \frac{du}{dx} - \frac{1}{x}\,u &= u^2 \end{aligned}$$

It resembles first order linear DE except the member on the right side. It is Bernoulli DE with solution

$$u = \frac{2x}{2c_1 - x^2}\,.$$

Let us rewrite u to y as we are used to (to avoid confusion) and solve the above Bernoulli DE.

$$y' - \frac{1}{x}\,y = y^2$$

We need to eliminate y^2 on the right side to transform the task to linear DE. That will be achieved by multiplying both sides by y^{-2}. We have learnt that we use substitution $u = y^{-1}$, then multiply DE by $u' = -y^{-2}$:

$$\begin{aligned} -y^{-2}y' + y^{-2}\,\frac{1}{x}\,y &= -y^{-2}y^2 \\ \frac{du}{dx} + \frac{1}{x}\,\frac{1}{y} &= -1 \\ \frac{du}{dx} + \frac{1}{x}\,u &= -1 \end{aligned}$$

We really got to linear DE, which has solution

$$u = -\frac{x}{2} + \frac{c_1}{x}\,.$$

The above linear DE is not difficult to solve. We can locate $P(x) = 1/x$, then integrating factor is $\mu(x) = e^{\int P(x)\,dx} = e^{\int 1/x\,dx} = x$. We multiply both sides of DE by integrating factor $\mu(x)$ and locate $\frac{d}{dx}\,\mu(x) \cdot u$ and finally integrate both sides.

$$\begin{aligned} x\,\frac{du}{dx} + u &= x(-1) \\ \frac{d}{dx}\,(x \cdot u) &= -x \\ xu &= -\frac{x^2}{2} + c_1 \\ u &= -\frac{x}{2} + \frac{c_1}{x}\,. \end{aligned}$$

Let us go back against substitution $u = y^{-1} = 1/y$ we have used in the beginning:

$$\begin{aligned} \frac{1}{y} &= -\frac{x}{2} + \frac{c_1}{x} \\ y &= \frac{2x}{2c_1 - x^2} \end{aligned}$$

After integrating we have general solution

$$y = -\log|2c_1 - x^2| + c_2\,.$$

Example

Solve nonlinear DE $y'' + (y')^3 y = 0$.

Again, let us use substitution $u = y'$.

$$u' + u^3 y = 0 \tag{62}$$

Variable x is independent and variable y is dependent. If we use $u' = du/dx$ again as in previous example, it will be not helpful: because it brings us to $du/dx + u^3 y = 0$. Let us use *chain rule*:

$$\begin{aligned} u = y' &= \frac{dy}{dx} \\ u' &= \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}\,y' = \frac{du}{dy}\,u \end{aligned}$$

Finally we have avoided dx and we can work on DE only in terms of u and y.

$$\begin{aligned} u\,\frac{du}{dy} + u^3 y &= 0 \\ \frac{u}{u^3}\,du + y\,dy &= 0 \qquad u \neq 0 \\ u^{-2} du + y\,dy &= 0 \qquad \text{(integrate both sides)} \\ -u^{-1} &= -\frac{y^2}{2} + c_1 \end{aligned} \tag{63}$$

From the steps above we got two solutions. We have laid $u \neq 0$, so we have to investigate also whether $u = 0$ is a solution. If we substitute $u = 0$ into (62) then it is confirmed that $u = 0$ is a solution as well. Thus we have two solutions:

$$u(y) = \frac{2}{y^2 + 2c_1} \qquad \text{and} \qquad u(y) = 0$$

Going back through substitution $u = dy/dx$ and integrate both sides:

$$\begin{aligned} \frac{dy}{dx} = \frac{2}{y^2 + 2c_1} &\implies \frac{y^3}{3} + 2c_1 y = 2x + c_2 \\ \frac{dy}{dx} = 0 &\implies y = c \end{aligned}$$

Example

Solve nonlinear DE $y^2 y'' - (y')^3 = 0$.

We have to use substitution $u = y'$. Then

$$y^2 u' - u^3 = 0. \tag{64}$$

Before we start to solve (64) we have to consider what is u' here. Variable u depends on x, it is du/dx. But that is not helpful since (64) becomes $y^2\,du/dx - u^3 = 0$ then. We have to use chain rule to eliminate dx.

$$u' = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}\,u$$

That helps us to eliminate dx and we can solve DE where u depends on y:

$$y^2 \frac{du}{dy} u - u^3 = 0$$
$$y^2 \frac{du}{dy} - u^2 = 0$$
$$y^2 du - u^2 dy = 0$$
$$\frac{1}{u^2} du - \frac{1}{y^2} dy = 0$$
$$-u^{-1} + y^{-1} = c_1$$

The DE (64) is solved, now we have to walk back against substitution $u = y' = dy/dx$ from the beginning.

$$-\frac{dx}{dy} + y^{-1} = dy$$
$$-dx = dy(c_1 - y^{-1})$$
$$-x = c_1 y - \log|y| + c_2$$

That can be a little improved to

$$\underline{\underline{\log|y| - c_1 y = x + c_2}}\,.$$

Example

Solve $y'' + 2y'/x + x^2 = 3$.

It can be observed that when this DE is multiplied by x^2, it becomes Cauchy-Euler DE, which has solutions based on x^m. Here it will be solved by substitution $u = y'$ and the solutions regardless the method have to be the same.

If you have a skill, you might be also able to find first solution $y_1 = c_1$ from associated homogeneous DE and then use *reduction of order* method to find solution $y_2 = c_2 x^{-1}$.

$$u' + \frac{2}{x} u + x^2 = 3$$
$$\frac{du}{dx} + \frac{2}{x} u = 3 - x^2$$

We got to linear DE of first order which has to be solved by integrating factor $\mu(x)$ with $P(x) = 2/x$. Note: if multiplied by x it can be also recognized and solved again as a Cauchy-Euler DE.

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}\,dx} = x^2$$

If we multiply DE by integrating factor $\mu(x)$ we anticipate that on the left side is $d/dx\; \mu(x)u$ to be found:

$$\frac{d}{dx} \mu(x)u = 3x^2 - x^4$$
$$\mu(x)u = \int (3x^2 - x^4)\,dx$$
$$x^2 u = x^3 - \frac{1}{5} x^5 + c_1$$
$$u = x - \frac{1}{5} x^3 + c_1 \frac{1}{x^2}$$

$$y' = x - \frac{1}{5} x^3 + c_1 \frac{1}{x^2}$$

So the final solution, which could be also obtained by other method, is

$$y = \frac{x^2}{2} - \frac{1}{20}x^4 - c_1 x^{-1} + c_2\,.$$

Taylor series solution for initial value problems

If initial values are given, nonlinear DE might be solvable by means of Taylor polynomial. The advantage of Taylor series is that the numerical solution converge fast.

First let us remind the Taylor polynomial:

$$y(x) = y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \frac{y'''(a)}{3!}(x-a)^3 + \dots + \frac{y^{(k)}(a)}{k!}(x-a)^k$$

If we are able to differentiate given function $y(x)$, then we are able to approximate such function by means of Taylor series. The series is made for the $x = a$, then the approximation works best around that given point.

The first members are given by initial values. The next members have to be expressed from DE.

Example

Solve $y'' = e^x$ with initial values $y(0) = 0,\ y'(0) = -1$.

We need to find solution $y(x)$. For that aim we may use Taylor polynomial and solve numerically around point $x = 0$, that means $a = 0$ when composing the series. The first and the second member of series is given by initial values. The third member is given by DE itself. To establish next members, we have to differentiate given DE to obtain $y''',\ y^{(4)}$ and so on.

Let us start with initial values:

$$y(x) = 0 + (-1)(x-0) + \dots$$

Now we may append the member $y''(x)$ which comes from DE as e^x:

$$y(x) = 0 + (-1)(x-0) + \frac{e^0}{2!}(x-0)^2 \dots$$

The next members come from further differentiating of y'', that means from differentiating e^x:

$$y(x) = 0 + (-1)(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3 + \frac{e^0}{4!}(x-0)^4 + \dots$$

or

$$y(x) = 0 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

If we use above series with 5 members the approximations of the solution at $x = \{-0.2; 0.2; 0.4\}$ are

$$\begin{aligned} y(-0.2) &= 0 - 0.2 + 0.02 - 0.001333 + 6.67\cdot 10^{-5} &&= +0.21873 \\ y(0.2) &= 0 - 0.2 + 0.02 + 0.001333 - 6.67\cdot 10^{-5} &&= -0.1786 \\ y(0.4) &= 0 - 0.4 + 0.08 + 0.016667 + 0.001667 &&= -0.30827 \end{aligned}$$

Analytical solution is $y(x) = c_1 + c_2 x + e^x$ or $y(x) = -1 - 2x + e^x$ (when solved with initial values) which brings the solutions:

$$\begin{aligned} y(-0.2) &= +0.21873 \\ y(0.2) &= -0.17860 \\ y(0.4) &= -0.30818 \end{aligned}$$

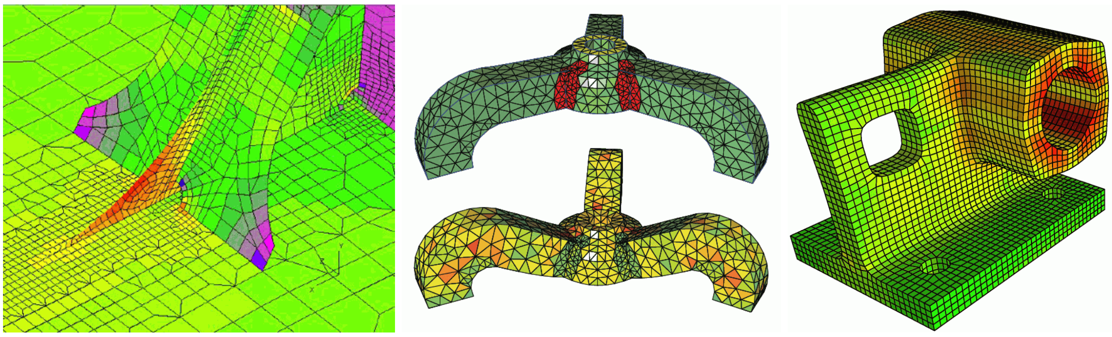
Modeling with higher order linear differential equations, initial values

In structural analysis for civil engineering we deal with tasks which are

- static in time and
- dynamic in time.

The first one is the most common because civil engineering projects usually serves without motions during the lifetime.

Some case might require dynamic analysis of building/structure. For example the effects of earthquake, winds, explosion, collision and so on. This kind of problems is usually modeled by means of software for FEM (finite element method). The structure is divided into finite number of bodies/elements which interact by means of springs.



The finite element method is a numerical technique to simulate and study many kinds of real-world problems. For example the structure is divided into finite elements and it leads to solving systems of equations, in case of dynamic behaviour into system of differential equations.

Such system internally leads into system of differential equations

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f},$$

where $\boldsymbol{M}$ represents mass, $\boldsymbol{C}$ damping, $\boldsymbol{K}$ stiffness and $\boldsymbol{f}$, $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, $\ddot{\boldsymbol{u}}$ are external force, displacement, velocity and acceleration (the latter four are functions of time).

Note: we will use rather symbol x instead of u for displacement in text below.

The solving leads to harmonic motions of frequency ω

$$\boldsymbol{u}(t) = \boldsymbol{U}\cos(\omega t - \alpha),$$

where $\boldsymbol{U}$ is a vector that collects amplitudes.

The task is then to compute eigenvectors and eigenvalues to find natural frequencies, to study resonances and their desired or undesired impacts.

For a body on a spring we recognize 3 basic cases:

1. Free undamped motion.
2. Free damped motion.
3. Driven motion (induced by external load).

Free undamped motion

Note: undamped = no resistance causing decreasing of amplitude. We will employ

- **Hookes's law** $F = ks$
  k is a property of the spring (spring constant, stiffness), s is displacement from unstretched position
- **Newton's second law** $F = ma$
  m is a mass (weight), g is acceleration

Let us study the case.

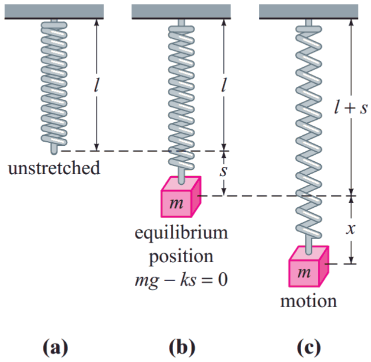


FIGURE 5.1.1 Spring/mass system

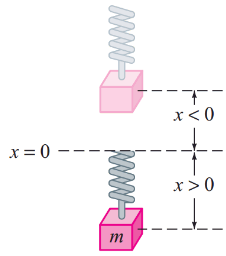


FIGURE 5.1.2 Direction below the equilibrium position is positive.

When the body on the spring is in equilibrium (no movement) then we can observe that force from the spring ks has the same magnitude as gravity force mg, i.e. $mg = ks$ or $mg - ks = 0$ (displacement down is positive). The system is in equilibrium, the force $mg - ks = 0$, thus no movement nor acceleration.

The troubles start when the system is not in equilibrium. Instead of ks the force from the spring is $k(s + x)$. The body is exposed to a force $mg - k(s + x)$, i.e.

$$m\, a(t) = mg - k(s + x)$$

and since we want to study motion x:

$$m\,\frac{d^2x}{dt^2} = mg - k(s + x) = (mg - ks) - kx.$$

It has been noticed from the equilibrium state that $mg - ks = 0$, so

$$m\,\frac{d^2x}{dt^2} = -kx. \tag{65}$$

Note that if the distance $s(t)$ [m] in time is known, then the speed $v(t)$ [m/s] can be derived. If the speed at any time is known, then the acceleration $a(t)$ $[m/s^2]$ can be derived.

$$v(t) = \frac{ds}{dt} = \frac{dx}{dt}\,, \qquad a(t) = \frac{d^2x}{dt^2}\,.$$

If we divide (65) by m, we get simple DE of second order

$$\begin{aligned} x'' + \frac{k}{m}\,x &= 0 \\ x'' + \omega^2 x &= 0, \qquad \omega^2 = \frac{k}{m} \\ m^2 + \omega^2 &= 0 \implies m = \{i\omega, -i\omega\} \end{aligned}$$

or

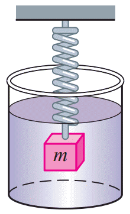
$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t. \tag{66}$$

Both sin() and cos() have a period of 2π, thus for the motion being described $2\pi = \omega t \implies T = 2\pi/\omega$ is the period.

In physics or dynamics, it is a common practice to work with rather simplier description of the motion in the form of $u(t) = U \cos(\omega t - \alpha)$, which can be readily converted from (66).

Free damped motion

The description (66) is unrealistic because in real world the movement of the body on spring will not last forever.



(a)

FIGURE 5.1.5 Damping device

We know from physics, that the faster the movement is, the stronger is damping force. Damping force F_c is proportional to speed ($F_c = cv$, where c is positive damping constant).

$$m\,\frac{d^2x}{dt^2} = -kx - c\,\frac{dx}{dt}\,. \tag{67}$$

The sign is negative, because damping force (also the spring) acts the opposite side against the motion.

If we divide (67) by m and bring some substitutions for convenience, we get

$$x'' + 2\lambda x' + \omega^2 x = 0, \qquad 2\lambda = \frac{c}{m}\,,\ \omega^2 = \frac{k}{m}\,,$$

$$m^2 + 2\lambda m + \omega^2 = 0 \implies \begin{cases} m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}, \\ m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}. \end{cases} \qquad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It depends on $\lambda^2 - \omega^2$ whether roots $m_1,\ m_2$ repeat, are real or imaginary:

- **Roots are real** when $\lambda^2 - \omega^2 > 0$
  The solution is $x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2-\omega^2}\cdot t} + c_2 e^{\sqrt{\lambda^2+\omega^2}\cdot t})$. The motion described by $x(t)$ has **no periodicity,** it is exponential function. The **system is overdamped.**
- **Roots repeat** when $\lambda^2 - \omega^2 = 0$
  The solution is $x(t) = c_1 e^{m_1 t} + c_2 t e^{m_2 t} = e^{-\lambda t}(c_1 + c_2 t)$. The motion is **critically damped**. There is no oscillatory motion but a slight change in parameters may result in oscillatory motion.
- **Roots are imaginary** when $\lambda^2 - \omega^2 < 0$
  Since $i^2 = -1$ the solution is $x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}\,t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\,t)$. The presence of sin() and cos() suggests that this case is **oscillatory (underdamped).**

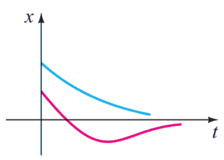


FIGURE 5.1.6 Motion of an overdamped system

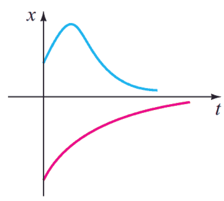


FIGURE 5.1.7 Motion of a critically damped system

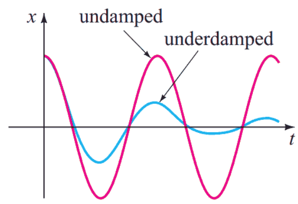


FIGURE 5.1.8 Motion of an underdamped system

All the above cases share the member $e^{-\lambda t}$ which represents the fact, that the amplitude of the motion is decreasing over time due to the damping.

Driven motion

In the two previous examples the motion comes from given initial velocity and/or given initial displacement. The more general case is with external force $f(t)$ involved:

$$m\,\frac{d^2x}{dt^2} = -kx - c\,\frac{dx}{dt} + f(t) \qquad \text{or}$$

$$m\,\frac{d^2x}{dt^2} + kx + c\,\frac{dx}{dt} = f(t) \tag{68}$$

DE (68) is nonhomogeneous. Function on the right side (in this case $f(t)$) is sometimes refered as a *driving* or *forcing* function of the system.

Example

A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time.

From the above information we can calculate spring constant k.

$$k = \frac{25.6}{0.7 - 0.5} = 128 \text{ N/m}$$

We have enough data to collect differential equation of the motion (see also (65)).

$$\begin{aligned} 2x'' &= -128x \\ x'' + 64x &= 0 \end{aligned}$$

Auxiliary equation is

$$m^2 + 64 = 0 \implies m = \{8i, -8i\} \qquad (\alpha = 0,\ \beta = 8)$$

And the solution is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

The task remains to find values of arbitrary constants. We know that at time $t = 0$ velocity $v = x' = 0$ and initial displacement from the equilibrium position is $x = +0.2$ m:

$$\begin{aligned} 0.2 &= c_1 \cos 8t + c_2 \sin 8t \\ 0 &= -8c_1 \sin 8t + 8c_2 \cos 8t \end{aligned}$$

$$\begin{aligned} 0.2 &= c_1 \cdot 1 + c_2 \cdot 0 \implies c_1 = 0.2 \\ 0 &= -8c_1 \cdot 0 + 8c_2 \cdot 1 \implies c_2 = 0 \end{aligned}$$

Then

$$\underline{\underline{x(t) = 0.2 \cos 8t}}$$

is the description of the position of the mass at any time.

Example

The same spring is immersed into fluid with damping constant c = 40 N.s/m. The motion starts from equilibrium position and is given initial velocity 0.6 m/s.

$$\begin{aligned} 2x'' &= -128x - 40x' \\ 2x'' + 40x' + 128x &= 0 \\ x'' + 20x' + 64x &= 0 \end{aligned}$$

Auxiliary equation is

$$m^2 + 20m + 64 = 0 \implies m = \{-4, -16\}$$

The solution is

$$x(t) = c_1 e^{-4t} + c_2 e^{-16t}$$

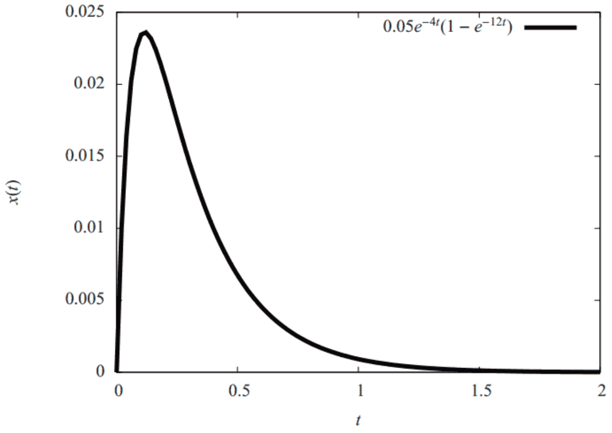
Now the task remains to find values of arbitrary constants $c_1,\ c_2$. At the time $t = 0$: $x = 0,\ x' = 0.6$.

$$\left.\begin{aligned} 0 &= c_1 e^0 + c_2 e^0 \\ 0.6 &= -4c_1 - 16c_2 \end{aligned}\right\} \implies c_1 = 0.05,\ c_2 = -0.05$$

Then

$$x(t) = 0.05e^{-4t} - 0.05e^{-16t} = 0.05e^{-4t}(1 - e^{-12t})$$

is the description of the position of the mass at any time.



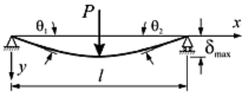
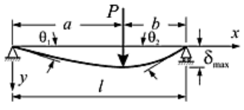
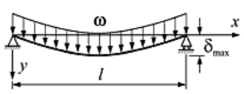
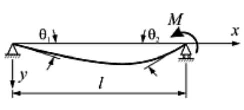
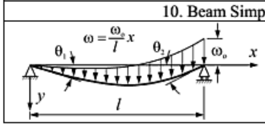
The motion of the body on the spring. Bottom axis is time t from 0 to 2 s. Left axis is x(t) from 0 to 0.25 m.

Modeling with higher order linear differential equations, boundary-value problems

Deflection of a Beam

In civil engineering, analyzing structures for their internal forces and deflections is one of the most important topic.

BEAM DEFLECTION FORMULAS

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load P at the center			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$y = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load P at any point			
	$\theta_1 = \frac{Pb(l^2 - b^2)}{6lEI}$ $\theta_2 = \frac{Pab(2l - b)}{6lEI}$	$y = \frac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + (l^2 - b^2)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \frac{Pb\left(l^2 - b^2\right)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{(l^2 - b^2)/3}$ $\delta = \frac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$
8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \frac{\omega l^3}{24EI}$	$y = \frac{\omega x}{24EI}\left(l^3 - 2lx^2 + x^3\right)$	$\delta_{max} = \frac{5\omega l^4}{384EI}$
9. Beam Simply Supported at Ends – Couple moment M at the right end			
	$\theta_1 = \frac{Ml}{6EI}$ $\theta_2 = \frac{Ml}{3EI}$	$y = \frac{Mlx}{6EI}\left(1 - \frac{x^2}{l^2}\right)$	$\delta_{max} = \frac{Ml^2}{9\sqrt{3}\,EI}$ at $x = \frac{l}{\sqrt{3}}$ $\delta = \frac{Ml^2}{16EI}$ at the center
10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta_1 = \frac{7\omega_o l^3}{360EI}$ $\theta_2 = \frac{\omega_o l^3}{45EI}$	$y = \frac{\omega_o x}{360lEI}\left(7l^4 - 10l^2x^2 + 3x^4\right)$	$\delta_{max} = 0.00652\frac{\omega_o l^4}{EI}$ at $x = 0.519l$ $\delta = 0.00651\frac{\omega_o l^4}{EI}$ at the center

Tables used in structural analysis by civil engineers

The above table depicts θ and y for several types of structures and/or their loads. The problem of deflections was observed and described by means of differential equations. In structure analysis we usually work either with precomputed results (see the table above) or we work routinelly with simple DE equations of higher order.

There are some rules or a guideline worth to mention. **The deflection y is a linear displacement measured from the beam's axis.** Angular displacement θ can be derived from deflection y. In structural analysis we typically start from load (either $w(x)$ or a single external force P), then we are able to compute internal shear forces $V(x)$ on the beam and internal moments $M(x)$ on the beam. From $M(x)$ we are able to get $\theta(x)$ by integration or we are able to get linear displacement $y(x)$ by double integration of $M(x)$.

1. The load $w(x)$ is typically given.
2. The internal shear forces $V(x)$ can be found from equilibrium of forces (and also $V(x) = -\int w(x)\,dx$).
3. The internal moments $M(x)$ can be found from equilibrium of forces (and also $M(x) = \int V(x)\,dx$).
4. The angular displacement θ can be found by integrating

$$\theta(x) = \frac{1}{EI}\int M(x)\,dx, \quad \text{because} \quad \frac{d\theta}{dx} = \frac{M}{EI}$$

.

5. The deflection y can be found by double integrating

$$y(x) = \frac{1}{EI}\iint M(x)\,dx, \quad \text{because} \quad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

.

In short we integrate $w(x) \to V(x) \to M(x) \to \theta(x) \to y(x)$. **The task is then to work on a boundary-value problem.** The boundary value problem is a differential equation with a set of additional restraints.

Example

*(The **first** case from the above table: simple beam with a concentrated load P in the middle)*
Assuming we have already computed moment function as $M(x) = Px/2$ for $0 < x < L/2$, find deflection $y(x)$ and $y_{\max}$ at $x = L/2$.

From the differential equation, describing deflection of the beam, we know, that we need to integrate $M(x)$ two times to get desired deflection.

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \implies y(x) = \frac{1}{EI}\iint M(x)\,dx \implies y(x)\cdot EI = \frac{Px^3}{12} + c_1x + c_2$$

The task remains to find constants $c_1,\ c_2$. These will be obtained by means of boundary value conditions.

It can be observed that

1. **at $x = 0$ the deflection is $y = 0$.** The second condition is
2. **at the middle of the span ($x = L/2$) the slope $\theta = 0$.**

Note that these are not initial values, since they are in different points. From the first condition we get

$$y(x)\cdot EI = \frac{Px^3}{12} + c_1x + c_2$$
$$0 = P\cdot 0 + c_1\cdot 0 + c_2 \implies c_2 = 0$$

For the second condition: we have to differentiate $y(x)$ and then we can get c_1.

$$y'(x)\cdot EI = \frac{Px^3}{12} + c_1$$
$$0 = \frac{P(\frac{L}{2})^3}{12} + c_1\frac{L}{2} \implies c_1 = -\frac{PL^2}{16}$$

So the deflection curve on the left half of the beam is

$$\underline{\underline{y(x)\cdot EI = \frac{Px^3}{12} - \frac{PL^2}{16}x}}$$

The maximum deflection at the middle of the span is

$$\underline{\underline{y_{\max}\cdot EI = \frac{P(\frac{L}{2})^3}{12} - \frac{PL^2}{16}\frac{L}{2} = -\frac{1}{48}\frac{P}{L^3}}}$$

So we obtained the same results which are written within the above table for civil engineers.

Example

*(The **third** case from the above table: simple beam with an uniformly distributed load w)*
Assuming we have already computed moment function as $M(x) = wLx/2 - x^2w/2$ for $0 < x < L$, find deflection $y(x)$ and $y_{\max}$ at $x = L/2$.

From the differential equation, describing deflection of the beam, we know, that we need to integrate $M(x)$ two times to get desired deflection.

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \implies y(x) = \frac{1}{EI}\iint M(x)\,dx \implies y(x)\cdot EI = \frac{wLx^3}{12} - \frac{x^4w}{24} + c_1x + c_2$$

The task remains to find constants $c_1,\ c_2$. These will be obtained by means of boundary value conditions.

It can be observed that

1. **at $x = 0$ the deflection is $y = 0$.** The second condition is
2. **at the end of the span ($x = L$) the deflection has to be also $y = 0$.**

Note that these are not initial values, since they are in different points. From the first condition we get

$$y(x) \cdot EI = \frac{wLx^3}{12} - \frac{x^4 w}{24} + c_1 x + c_2$$
$$0 = 0 - 0 + c_1 \cdot 0 + c_2 \implies c_2 = 0$$

From the second condition we can get c_1.

$$y(x) \cdot EI = \frac{wLx^3}{12} - \frac{x^4 w}{24} + c_1 x$$
$$0 = \frac{wLL^3}{12} - \frac{L^4 w}{24} + c_1 L \implies c_1 = -\frac{wl^3}{24}$$

So the deflection curve of the beam is

$$\underline{\underline{y(x) \cdot EI = \frac{wLx^3}{12} - \frac{x^4 w}{24} - \frac{wL^3}{24} x}}$$

The maximum deflection at the middle of the span $\left(x = L/2\right)$ is

$$\underline{\underline{y_{\max} \cdot EI = \frac{wL^4}{12 \cdot 8} - \frac{L^4 w}{24 \cdot 16} - \frac{wL^4}{24 \cdot 2} = -\frac{5}{384} wL^4}}$$

So we obtained the same results which are written within the above table for civil engineers.

Modeling with higher order nonlinear differential equations

Rocket motion, second cosmic speed

Example

Use Newton's second law and Newton's law of universal gravitation to find out the second cosmic speed (i.e. the escape speed).

There are many approaches how to determine second cosmic speed. We will describe the motion $y(t)$ by means of differential equation in order to demonstrate higher order nonlinear DE and then use to determine second cosmic speed..

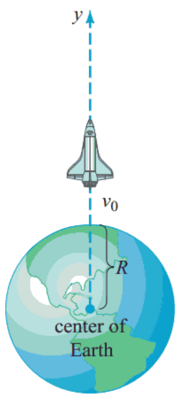


FIGURE 5.3.5 Distance to rocket is large compared to R.

Because distance is large compared to R, gravitational acceleration can not be considered as a constant.

Newton's second law says that when the sum of forces on a body is zero, then acceleration of the body is zero. Otherwise the net force is proportional to acceleration of the body.

The position $y(t)$ of the body can be described by DE

$$m\frac{d^2y}{dt^2} = -mg$$
$$\frac{d^2y}{dt^2} = -g$$

So mg above is force of gravity. The minus sign is necessary: we have taken the upward direction as positive. But earth's force of gravity does not act in such way.

Assuming we are able to solve DE including its constants (from initial values), then we have a description of motion as function of t. For a rocket motion, constant g can not be viewed as a constant. We have to consider that far from the earth it is a function of y (i.e. $g = f(y)$). The two bodies are attracted by a force

$$F = k\frac{mM}{R^2} = k\frac{mM}{y^2}$$

so we can rewrite gravitational acceleration as a function of distance:

$$\frac{d^2y}{dt^2} = -k\frac{M}{y^2}.$$

The constants k, M are unwelcome and we can eliminated them. If $y = R$, then acceleration has a value of g or $g = kM/R^2 \implies k = gR^2/M$.

$$\frac{d^2y}{dt^2} = -g\frac{R^2}{y^2}$$
$$y''y^2 = -gR^2$$

It is nonlinear DE of the second order. Finally we will not solve for y but only for speed v. Velocity $v = y'$.

$$v' y^2 = -gR^2, \qquad v = y' = \frac{dy}{dt}, \qquad v' = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy} v$$

$$\begin{aligned} \frac{dv}{dy} v\, y^2 &= -gR^2 \\ dv\, v &= -gR^2 \frac{1}{y^2} dy \\ \frac{v^2}{2} &= gR^2 \frac{1}{y} + c_1 \\ v^2 &= 2gR^2 \frac{1}{y} + 2c_1 \\ v &= \sqrt{2gR^2 \frac{1}{y} + 2c_1} \end{aligned}$$

We have employed chain rule in order to solve DE. So far we have speed v as a function of distance y. The constant c_1 can be determined from the initial value: on the surface of the earth ($y = R$) the body has an initial speed v_0:

$$\begin{aligned} v_0 &= \sqrt{2gR^2 \frac{1}{R} + 2c_1} \\ v_0^2 &= 2gR + 2c_1 \\ c_1 &= \frac{v_0^2}{2} - gR \end{aligned}$$

Thus we have

$$v(y) = \sqrt{{\color{red}2gR^2 \frac{1}{y}} + 2\left({\color{blue}\frac{v_0^2}{2} - gR}\right)}$$

Now we are back to the question from the beginning. What is the second cosmic speed? What *initial speed* the body needs to be given, that the gravitational forces are not able to call the body back to the earth.

At any point ($y \to \infty$) on the body's trip the body has not to stop. The speed $v(y)$ have to be always greater than zero. At such large distance ($y \to \infty$) the member $2gR^2/y$ is zero. Then the speed $v(y)$ is greater than zero if

$$\frac{v_0^2}{2} > gR \implies \underline{\underline{v_0 > \sqrt{2gR} = 11.2 \text{ km/s}}}$$

So, if you want to escape the earth, you have to do it pretty quickly. Note: it can be shown that the second cosmic velocity does not require direction perpendicular to earth surface.

Series solution of linear DE

As was pointed before

- some DE are difficult to solve or can not be solved analytically;
- some of the solutions consist of functions which are not elementary functions.

Example: The purpose is to solve **linear** DE

$$y'' - (x+1)y' + x^2 y = x$$

by means of infinite series. In this case the solution is

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \dots \qquad \text{for} - \infty < x < \infty.$$

Power series

A series of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{69}$$

where $c_0,\ c_1, \dots,\ a$ are constants and x is variable, is called a power series. Such power series is centered at a.

Convergence of power series

We study whether the above **sum** approaches some given number. Then the series converges. A power series may

1. Converge *only* for a single value $x = a$.
2. Converge absolutely for values x near value a, but not for other values. That means
   - there is and interval $(a-h; a+h)$ within which x converges
   - and diverges for other values.
   - At the end points $x \pm h$ may either converge or diverge.
3. Converge absolutely for all values of x, i.e. for $-\infty < x < \infty$.

Ratio test

Convergence of power series can be determined by the ratio test. It states that the series $u_1 + u_2 + u_3 + \dots + u_n + \dots$ **converges** absolutely if

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = L \qquad \text{and} \quad L < 1.$$

If $L > 1$ then the series diverges. If $L = 1$ the test is inconclusive.

Example

Determine whether series $1 + 1!\,x + 2!\,x^2 + 3!\,x^3 + \dots + n!\,x^n + \dots$ converges.

Here

$$u_n = n! \cdot x^n$$
$$u_{n+1} = (n+1)! \cdot x^{n+1}$$

Therefore

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+1)!\,x^{n+1}}{n!\,x^n} \right| = (n+1)|x|$$

The ratio between both members u_{n+1} and u_n grows with $n \to \infty$: $L = (n+1)|x| \to \infty$ as $n \to \infty$. Since it is not valid that $L < 1$, the series converges only for $x = 0$.

Example

Determine whether series $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \frac{1}{n}x^n$ converges.

Here

$$u_n = \frac{1}{n}x^n,$$
$$u_{n+1} = \frac{1}{n+1}x^{n+1}$$
$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = |x| = L$$

A series converges if $L < 1$. In our case the series converges for $-1 < x < 1$.

Note: the test is not conclusive at endpoints ± 1: in this case for $x = 1$ diverges, for $x = -1$ converges (check the series itself).

Shifting the summation index of power series

To solve DE by power series, it is crucial to be able to simplify two series into one.

Example

Simplify $\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$

The important observation is that both series are not expressed from x^n. Instead of that they use x^{n-1}, x^{n+1}. It is desired to have both in the form of x^n.

$$\sum_{n=1}^{\infty} 2nc_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1)c_{n+1}x^n$$
$$\sum_{n=0}^{\infty} 6c_n x^{n+1} = \sum_{n=1}^{\infty} 6c_{n-1}x^n$$

Then sum of above series is

$$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} = \sum_{n=0}^{\infty} 2(n+1)c_{n+1} \cdot x^n + \sum_{n=1}^{\infty} 6c_{n-1} \cdot x^n =$$
$$= 2c_1 + \sum_{n=1}^{\infty} x^n(2(n+1)c_{n+1} + 6c_{n-1})$$

A power series defines a function

If power series converges on an interval I, then the power series (69) defines a function $f(x)$ which is continuous, differentiable and integrable on the interval:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$
$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \ldots$$
$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$$

Analytic at point

Any function is said to be analytic at a point a if it can be represented by a power series in $x - a$.

Example: function such as e^x, $\sin x$, $\log\ x$ can be represented by Taylor series. These functions are analytic.

If a function is analytic at point, it can be replaced either by Taylor series (70) or Maclaurin series (71), which are both power series.

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots \tag{70}$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\dots \tag{71}$$

Ordinary and singular points

A point $x = x_0$ is called ordinary point of the DE

$$y^{(n)}+F_{n-1}(x)y^{(n-1)}+\dots+F_1(x)y'+F_0(x)y = Q(x)$$

if each function $F_0,\ F_1,\ \dots,\ F_{n-1}$ and Q is analytic at $x = x_0$. A point $x = x_0$ is called singular if one or more of the functions $F_0(x),\ \dots,\ F_{n-1}(x),\ Q(x)$ are not analytic at $x = x_0$.

Example

DE $(x-1)y''+\frac{1}{x}y'-2y=0$ has two singularities: at $x=1$ and at $x=0$.

If we divide DE by $(x-1)$, we get

$$y''+\frac{1}{x(x-1)}y'-\frac{2}{x-1}y=0 \implies \begin{cases} F_1(x)=\dfrac{1}{x(x-1)} \\ F_0(x)=-\dfrac{2}{x-1}\end{cases}$$

Both $F_0,\ F_1$ are singular, are not analytic and can not be described by power series.

Existence of solution

If $x = x_0$ is an ordinary point of the DE, we can always find solutions in the form of power series centered at x_0. A power series solution converges at least on interval $|x-x_0| < R$, where R is the distance from the closest singular point.

Solving DE with power series

Note: in finding a series solution of a differential equation, it will usually not be easy to write the general form of the series. In fact, it will, in most cases, be very difficult if not impossible.

Example

Solve DE $(x-1)y''-xy'+y=0$, $y(0)=-2$, $y'(0)=6$.

First, let us solve by successive differentiation using Maclaurin polynomial.

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\dots$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = -2+6x+\cdots$$

The first two members were taken from initial values. The next members will be computed by successive differentiations:

$$
\begin{aligned}
f''(0): &\quad (x-1)y''-xy'+y=0 \implies y''(x=0,y=-2,y'=6)=-2\\
f'''(0): &\quad (x-1)y'''+y''-xy''-y'+y'=0\\
&\quad (x-1)y'''+y''-xy''=0 \implies y'''(x=0,y'=6,y''=-2)=-2\\
f^{(4)}(0): &\quad (x-1)y^{(4)}+y'''+y'''-xy'''-y''=0\\
&\quad (x-1)y^{(4)}+2y'''-xy'''-y''=0 \implies y^{(4)}(x=0,y''=-2,y'''=-2)=-2\\
f^{(5)}(0): &\quad (x-1)y^{(5)}+y^{(4)}+2y^{(4)}-xy^{(4)}-y'''-y'''=0\\
&\quad (x-1)y^{(5)}+3y^{(4)}-xy^{(4)}-2y'''=0 \implies y^{(5)}(x=0,y'''=-2,y^{(4)}=-2)=-2
\end{aligned}
$$

From values above we may form the series for solution:

$$y=f(x)=-2+6x-2\frac{x^2}{2!}-2\frac{x^3}{3!}-2\frac{x^4}{4!}-2\frac{x^5}{5!}+\dots$$

It can be noticed that the solution does not involve any arbitrary constant, because it is particular solution for given initial values.

Example

Solve DE $(x-1)y''-xy'+y=0$, $y(0)=-2$ and $y'(0)=6$.

The same example as above but now solved by power series **(undetermined coefficients approach).**

Let us approximate solution by

$$y=\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+c_3x^3+\dots$$

Then

$$
\begin{aligned}
y'&=\sum_{n=1}^{\infty}c_nnx^{n-1}=c_1+2c_2x+3c_3x^2+\dots\\
y''&=\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}=2c_2+6c_3x+12c_4x^2+\cdots
\end{aligned}
$$

Let us substitute $y,\ y',\ y''$ into DE:

$$
\begin{aligned}
(x-1)\Big[\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}\Big]-x\Big[\sum_{n=1}^{\infty}c_nnx^{n-1}\Big]+\Big[\sum_{n=0}^{\infty}c_nx^n\Big]=0\\
\Big[\sum_{n=2}^{\infty}c_nn(n-1)x^{n-1}-\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}\Big]-\Big[\sum_{n=1}^{\infty}c_nnx^n\Big]+\Big[\sum_{n=0}^{\infty}c_nx^n\Big]=0
\end{aligned}
$$

We need to shift the summation index in order to work with the series in a convenient way, i.e. all sums have to work with x^n:

$$\Big[\sum_{n=1}^{\infty}c_{n+1}(n+1)nx^n-\sum_{n=0}^{\infty}c_{n+2}(n+2)(n+1)x^n\Big]-\Big[\sum_{n=1}^{\infty}c_nnx^n\Big]+\Big[\sum_{n=0}^{\infty}c_nx^n\Big]=0$$

Let us collect coefficients $c_0,\ c_1,\ c_2,\ c_3,\ \dots$ for $1,\ x,\ x^2,\ x^3,\dots$. We are solving second order DE, which is linear. So we will have two independent solutions, which are distinguished by arbitrary constants $c_0,\ c_1$. The other constants have to be evaluated in terms of $c_0,\ c_1$.

$$n=0: \quad 1(-c_2\cdot 2\cdot 1+c_0)=0 \implies c_2=\frac{1}{2}\,c_0$$

$$n=1: \quad x(c_2\cdot 2\cdot 1-c_3\cdot 3\cdot 2-c_1\cdot 1+c_1)=0 \implies c_3=\frac{1}{3}\,c_2=\frac{1}{6}\,c_0$$

$$n=2: \quad x^2(c_3\cdot 3\cdot 2-c_4\cdot 4\cdot 3-c_2\cdot 2+c_2)=0 \implies c_4=\frac{c_2-6c_3}{-12}=\frac{1}{24}\,c_0$$

$$n=3: \quad x^3(c_4\cdot 4\cdot 3-c_5\cdot 5\cdot 4-c_3\cdot 3+c_3)=0 \implies c_5=\frac{2c_3-12c_4}{-20}=\frac{1}{120}\,c_0$$

From coefficients found so far we can form power series of y:

$$y=c_0+c_1x+\frac{1}{2}\,c_0x^2+\frac{1}{6}\,c_0x^3+\frac{1}{24}\,c_0x^4+\frac{1}{120}\,c_0x^5+\dots$$

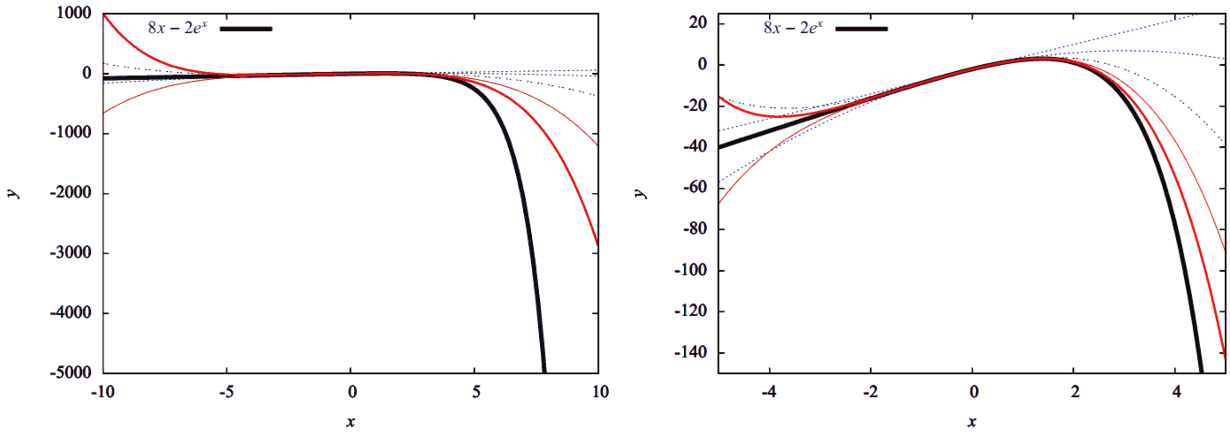
You can notice that we have two arbitrary constants $(c_0,\ c_1)$ within the above general solution y. And that implies there are two solutions $y_1(x)=c_1f_1(x)$ and $y_2(x)=c_2f_2(x)$. And that could be expected since the DE to solve is not nonlinear. The first solution is an infinite series with c_0 and the second solution is a line c_1x. That complies with the analyticial solution of the given DE, which is $y=c_1x+c_2e^x$.

From initial values we will determinate constants to express particular solution. $y(0)=-2 \implies c_0=-2$, $y'(0)=6 \implies c_1=6$. Then

$$y=-2+6x-x^2-\frac{1}{3}\,x^3-\frac{1}{12}\,x^4-\frac{1}{60}\,x^5+\dots \quad \text{or}$$

$$y=-2+6x-2\,\frac{x^2}{2!}-2\,\frac{x^3}{3!}-2\,\frac{x^4}{4!}-2\,\frac{x^5}{5!}+\dots$$

We have reached the same solution as within previous example.



Series (2, 3, 4, 5, 6 members of infinite series involved) around $x=0$ compared to analytical solution (in black). The second chart is a detail from the first one.

Example

Solve nonlinear DE $y''=xy-(y')^2$, $y(0)=2$ and $y'(0)=1$ by power series (undetermined coefficients approach).

We will by finding the general solution first and then the initial values will be used to evaluate $c_0,\ c_1$.

We can use the power series for $y,\ y',\ y''$ from the last example.

$$y=\sum_{n=0}^{\infty}c_nx^n, \quad y'=\sum_{n=1}^{\infty}c_nnx^{n-1}, \quad y''=\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}$$

and substitute them into DE.

$$\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}=x\sum_{n=0}^{\infty}c_nx^n-\Big[\sum_{n=1}^{\infty}c_nnx^{n-1}\Big]^2$$

It would be quite difficult to work with the above series if they are not based on x^n. They can be rewritten to

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n = \sum_{n=1}^{\infty} c_{n-1}x^n - \Big[\sum_{n=0}^{\infty} c_{n+1}(n+1)x^n\Big]^2$$

Note that the last series is squared, which brings us some more work. To handle the power, it is useful to list its members:

$$\Big[\sum_{n=0}^{\infty} c_{n+1}(n+1)x^n\Big]^2 = (c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots)^2$$

Now we do not need to square the series itself. But we need to pick up coefficients for each $1,\ x,\ x^2,\ \ldots$ For example we want to pick up coefficient of x^2. Now imagine there is the series multiplied by itself:

$$\Big[\sum_{n=0}^{\infty} c_{n+1}(n+1)x^n\Big]^2 = (c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots)\cdot$$
$$\cdot\,(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots)$$

Then for x^2: $(c_1 \cdot 3c_3x^2) + (3c_3x^2 \cdot c_1) + (2c_2x)^2$.

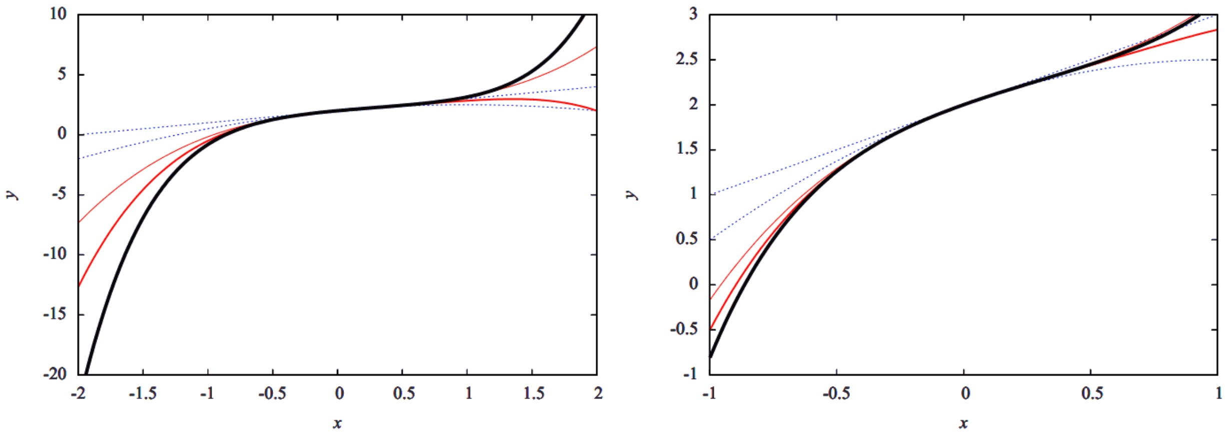
Now we can **use method of undetermined coefficients for $1,\ x,\ x^2,\ \ldots$.**

$$n=0: \quad 1(c_2 \cdot 2 = -c_1^2) = 0 \implies c_2 = \frac{c_1^2}{-2}$$
$$n=1: \quad x(c_3 \cdot 6 = c_0 - 2c_1c_2 \cdot 2) = 0 \implies c_3 = \frac{c_0 - 4c_1c_2}{6}$$
$$n=2: \quad x^2(c_4 \cdot 12 = c_1 - (3c_1c_3 \cdot 2 + 4c_2^2)) = 0 \implies c_4 = \frac{c_1 - (6c_1c_3 + 4c_2^2)}{12}$$
$$n=3: \quad x^3(c_5 \cdot 20 = c_2 - (4c_1c_4 \cdot 2 + c_2c_3 \cdot 6 \cdot 2)) = 0 \implies c_5 = \frac{c_2 - (8c_1c_4 + 12c_2c_3)}{20}$$

We have collected coefficients c_2 to c_5 to form a power series of solution. Since it is the second order DE, we should expect two arbitrary constants $c_0,\ c_1$. The others should be expressed in terms of $c_0,\ c_1$ and that is what can be confirmed by observing c_2 to c_5. If you observe c_3 you will notice that this time we can not separate solutions using c_0 and c_1 as in the previous example. The both solutions are somehow mixed. The reason is that the DE being solved is nonlinear.

Finally from given initial values $y(x=0)=2$ and $y'(x=0)=1$: $c_0=2, c_1=1$. Then the other constants can be evaluated as $c_2=-1/2$, $c_3=2/3$, $c_4=-1/3$, $c_5=37/120$.

$$\underline{\underline{y(x) = 2 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{37}{120}x^5 + \ldots}}$$



Series (2, 3, 4, 5, 6 members of infinite series involved) around $x=0$. No analytical solution drawn. The second chart is a detail from the first one.

Series solution of linear DE

Solution at singular point

It was explained in the last chapter that we have to analyse first whether the point is ordinary or singular. In the case the point is *ordinary*, we can find solution around that point by power series. The solution around singular points has been left to explain.

For **example** DE

$$(x-1)^2x^4y'' + 2(x-1)xy' - y = 0$$

has two singular points 0 and 1. If we try to find solution of DE at singular points *by successive differentiation* in the form of power series

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

we would run into trouble. Since $y''(0)$ is not defined at singular point, also $c_2 = y''(0)/2!$ (from Taylor series) does not exist. Moreover, if we try to use method of undetermined coefficients to determine $a_0,\ a_1,\ \ldots,$ they will be all zero. That is because after division by x^4 the coefficient of y' becomes not analytic at $x = 0$ and can not be expressed by Maclaurin series (we can not collect members of Maclaurin series at zero for a function $1/x$ or $1/x^3$.

If power series for y is modified by method of Frobenius, then—at regular singularity points—the solution can be expressed. Now we owe to divide singular points into

1. **regular singular points** (solution can be found by Frobenius method) and
2. **irregular singular points** (the problem of finding the series is too difficult to discuss here).

Regular and irregular singular points

Let us simplify the discussion only to the second order differential equations in the standard form of

$$y'' + P(x)y' + Q(x)y = 0. \tag{72}$$

Let us **multiply** functions

1. $P(x)$ by $(x-x_0)$,
2. $Q(x)$ by $(x-x_0)^2$.

After that,

1. if *each* member $P(x)\cdot(x-x_0)$ and $Q(x)\cdot(x-x_0)^2$ is analytic at $x = x_0$ then the singular point is regular.
2. Otherwise point is irregular singularity of DE.

Example

Determine the singularities of DE $(x-1)^3x^2y'' - 2(x-1)xy' - 3y = 0$.

First let us bring DE into standard form (72).

$$y'' - \frac{2}{x(x-1)^2}y' - \frac{3}{(x-1)^3x^2}y = 0$$

Two singularities are observed. At $x = \{0, 1\}$.

1. According to the rule, for the singularity at $x = 0$ we have to multiply $P(x) = \frac{-2}{x(x-1)^2}$ by x and $Q(x) = \frac{-3}{(x-1)^3x^2}$ by x^2. Then both members are analytic so the DE is a regular singularity at $x = 0$.
2. For the case at $x = 1$, if we multiply $P(x)$ by $(x-1)$ and $Q(x)$ by $(x-1)^2$, these members do not become analytic. So at $x = 1$ DE is irregular singularity.

Example

Determine the singularities of DE $(x-1)^2x^4y'' + 2(x-1)xy' - y = 0$.

First let us bring DE into standard form (72).

$$y'' + \frac{2}{(x-1)x^3}y' - \frac{1}{(x-1)^2x^4}y = 0$$

Two singularities are observed. At $x = \{0, 1\}$.

1. According to the rule, for the singularity at $x = 0$ we have to multiply $P(x) = \frac{2}{(x-1)x^3}$ by x and $Q(x) = \frac{-1}{(x-1)^2x^4}$ by x^2. Then both members are still not analytic so the DE is a irregular singularity at $x = 0$.
2. For the case at $x = 1$, if we multiply $P(x)$ by $(x-1)$ and $Q(x)$ by $(x-1)^2$, these members become analytic. So at $x = 1$ DE is regular singularity.

Method of Frobenius

If $x = x_0$ is a regular singularity then the series solution at such point will be found in the form of Frobenius series

$$y = (x-x_0)^r\,[c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \dots] \quad \text{or} \tag{73}$$
$$y = (x-x_0)^r\sum_{n=0}^{\infty} c_n(x-x_0)^n$$

For $r = 0$ or a positive integer the series (73) becomes Taylor series. A Frobenius therefore, includes the Taylor series as a special case.

Indicial equation and indicial roots

When solving by means of Frobenius series, one has to deal with **indicial equation** and its roots $r_1,\ r_2$. It will be demonstrated below on example, that for the second order DE we have two roots to expect. Each root is an origin of one of the solutions of DE then.

It might happen then that

1. **Roots are distinct, their difference not equal to an integer** (e.g. 1, 2, 3, ...)
   Each root $r_1,\ r_2$ is an origin of a series forming a solution.
2. **Roots are distinct, their difference equal to an integer**
   In some cases each root is an origin of a series solution, in other cases there will be only one series $y_1(x)$ and the other solution $y_2(x)$ has to be determined as a logarithmic solution from $y_1(x)$ and such logarithmic solution has limited applications.
3. **Roots are the same**
   There will be only one series $y_1(x)$ and the other solution $y_2(x)$ has to be determined as a logarithmic solution from y_1 (and has limited applications).

Example

Find solution of DE $x^2y'' + x(x+\frac{1}{2})y' + xy = 0$ at regular singularity.

First, if we bring DE into standard form (72) and apply rules to test singularities, it will be shown that $x = 0$ is a regular singularity. We will solve given DE in $x = 0$ by Frobenius series.

According to the Frobenius method, the solution will be found in the form

$$y = (x-x_0)^r\,[c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \dots$$

where two roots $r_1,\ r_2$ has to be evaluated. The coefficient c_0 is an arbitrary constant and the other constants will be expressed in terms of c_0. Thus we might expect two independent solutions.

Let us prepare $y,\ y',\ y''$ at $x_0 = 0$:

$$y = x^r(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots)$$
$$y' = c_0rx^{r-1} + c_1(r+1)x^r + c_2(r+2)x^{r+1} + \dots$$
$$y'' = c_0r(r-1)x^{r-2} + c_1(r+1)rx^{r-1} + c_2(r+2)(r+1)x^r + \dots$$

Let us substitute $y,\ y',\ y''$ into given DE

$$(c_0r(r-1)x^r + c_1(r+1)rx^{r+1} + c_2(r+2)(r+1)x^{r+2} \ + \ \dots) \ + \tag{74}$$
$$+(c_0rx^{r+1} + c_1(r+1)x^{r+2} + c_2(r+2)x^{r+3} + \dots \ + \tag{75}$$
$$+\frac{1}{2}c_0rx^r + \frac{1}{2}c_1(r+1)x^{r+1} + \frac{1}{2}c_2(r+2)x^{r+2} \ + \ \dots) \ + \tag{76}$$
$$+(c_0x^{r+1} + c_1x^{r+2} + c_2x^{r+3} \ + \ \dots) \ = \ 0 \tag{77}$$

So above is given DE written be means of infinite series. On (74) is x^2y'', on (75) is x^2y', on (76) is $\frac{1}{2}xy'$, on (77) is xy.

Now we have to use the analogy method to that method of undetermined coefficients (note that for any x^n there is always zero on the right side). Let us collect like powers of x from the above equation:

$$x^r \ (c_0r(r-1) + \frac{1}{2}c_0r) = 0 \tag{78}$$
$$x^{r+1} \ (c_1(r+1)r + c_0r + \frac{1}{2}c_1(r+1) + c_0) = 0 \tag{79}$$
$$x^{r+2} \ (c_2(r+2)(r+1) + c_1(r+1) + \frac{1}{2}c_2(r+2) + c_1) = 0$$
$$x^{r+3} \ (c_3(r+3)(r+2) + c_2(r+2) + \frac{1}{2}c_3(r+3) + c_2) = 0$$

Note that the first equation is used for a special purpose, i.e. to determine roots r: the equation (78) can be rewritten to $x^r c_0(r(r-1) + \frac{1}{2}r) = 0$ and is called **indicial equation**. We have no reason to expect that the first coefficients c_0 of series have to be zero. Then to find $r_1,\ r_2$:

$$(r(r-1) + \frac{1}{2}r) = 0 \implies r = \{0, \frac{1}{2}\}.$$

The first solution y_1 is based on $r = 0$. From (79) we can derive a recursion formula for coefficient c_n when $r = 0$:

$$c_nn(n-1) + c_{n-1}(n-1) + \frac{1}{2}c_nn + c_{n-1} = 0 \implies$$
$$\implies c_n = -c_{n-1}\frac{1}{n-1/2}$$

And from the recursive rule forming coefficients it is possible to form solution y_1:

$$y_1(x) = c_0(1 - 2x + \frac{4}{3}x^2 - \frac{8}{15}x^3 + \dots)$$

(If we use recursive formula of c_n, then e.g. for $c_0 = 1 \implies c_1 = -2$, for $c_1 = -2 \implies c_2 = \frac{4}{3}$ and so on.)

The second solution y_2 is based on $r = 1/2$. From (79) we can derive a recursion formula for coefficient c_n when $r = 1/2$:

$$c_n(\frac{1}{2} + n)(\frac{1}{2} + (n-1)) + c_{n-1}(\frac{1}{2} + (n-1)) + \frac{1}{2}c_n(\frac{1}{2} + n) + c_{n-1} = 0 \implies$$
$$\implies c_n = -c_{n-1}\frac{1}{n}$$

And if we use the last formula to determine coefficients c_n of the second solution we get

$$y_2(x) = c_0(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots).$$

The general solution expressed as a power series is then a combination of y_1 and y_2.

$$y(x) = C_0(1 - 2x + \frac{4}{3}x^2 - \frac{8}{15}x^3 + \ldots) + C_1(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots)$$

Example

Find solution of DE $x^2y'' - 3xy' + 4(x+1)y = 0$ at regular singularity.

When the DE is converted into basic form, it can be observed that there is a regular singularity at $x = 0$:

$$y'' - \frac{3y'}{x} + 4(\frac{x+1}{x^2})y = 0.$$

We expect the solution y in Frobenius form (73). Like in previous example, let us prepare $y', \ y''$

$$\begin{aligned} y &= x^r(c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots) \\ y' &= rx^{r-1}c_1 + (r+1)x^rc_2 + (r+2)x^{r+1}c_3 + \ldots \\ y'' &= r(r-1)x^{r-2}c_1 + (r+1)rx^{r-1}c_2 + (r+2)(r+1)x^rc_3 + \ldots \end{aligned}$$

and substitute them into DE in basic form:

$$\begin{aligned} &r(r-1)x^{r-2}c_1 + (r+1)rx^{r-1}c_2 + (r+2)(r+1)x^rc_3 + \cdots - \\ &-3\,\frac{1}{x}\,\left(rx^{r-1}c_1 + (r+1)x^rc_2 + (r+2)x^{r+1}c_3\right)\cdots + \\ &+4\,\frac{1}{x}\,\left(x^r(c_1 + c_2x + c_3x^2 + c_4x^3)\right) + \\ &+4\,\frac{1}{x^2}\,\left(x^r(c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots)\right) = 0 \end{aligned}$$

Now let us use the method of undetermined coefficients, first let us consider only the first possible term x^{r-2} in order to find indicial equation and its roots $r_1, \ r_2$:

$$x^{r-2}\,[r(r-1)c_1 - 3rc_1 + 4c_1] = x^{r-2}\left[c_1(r^2 - 4r + 4)\right] \implies r = \{2, \ 2\}$$

The roots are the same. We will evaluate only the first solution (for $\boldsymbol{r_1 = 2}$), the other solution, which is logarithmic, we are going to leave unexpressed.

$$\begin{aligned} x^{r-1}\,&[(r+1)rc_2 - 3(r+1)c_2 + 4c_1 + 4c_2] = \\ &= x\,[3\cdot 2c_2 - 3\cdot 3c_2 + 4c_1 + 4c_2] \implies c_2 = -4c_1 \end{aligned}$$

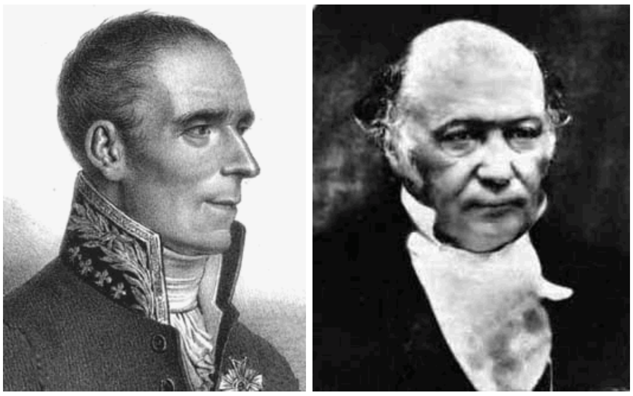
$$\begin{aligned} x^r\,&[(r+2)(r+1)c_3 - 3(r+2)c_3 + 4c_2 + 4c_3] = \\ &= x^2\,[4\cdot 3c_3 - 3c_3\cdot 4 + 4c_2 + 4c_3] \implies c_3 = -c_2 \end{aligned}$$

$$\begin{aligned} x^{r+1}\,&[(r+3)(r+2)c_4 - 3(r+3)c_4 + 4c_3 + 4c_4] = \\ &= x^3\,[5\cdot 4c_4 - 3c_4\cdot 5 + 4c_3 + 4c_4] \implies c_4 = -\frac{4}{9}\,c_3 \end{aligned}$$

And so on. Since we hav evaluated the coefficients, we are able to write the series:

$$y = c_1x^2(1 - 4x + 4x^2 - \frac{16}{9}x^3 + \ldots).$$

Laplace transform



Pierre-Simon Marquis de Laplace (1749-1827) and William Rowan Hamilton (1805-1865)
Sometimes we might feel some difficulties to keep the pace with the theories covered. That would be no surprise since they were discovered by geniuses of their age.

Pierre-Simon Marquis de Laplace (1749-1827) France (estimated IQ of 190)

Laplace advanced the nebular hypothesis of solar system origin, and was the first to conceive the concept of black holes. His other accomplishments in physics include theories about the speed of sound and surface tension. Laplace viewed mathematics as just a tool for developing his physical theories.

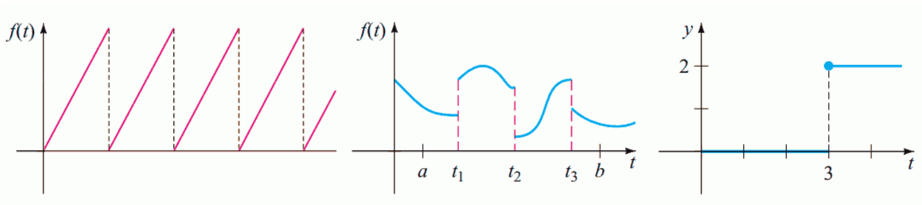
William Rowan Hamilton (1805-1865) Ireland (estimated IQ of 160—170)

Hamilton was a childhood prodigy. At age of 5 mastered Latin, Greek and Hebrew. Home-schooled and self-taught, **he started as a student of languages and literature. By the time he was 13, the future mathematician knew 13 different languages,** including Sanskrit, Persian, Italian, Arabic, Syriac and Indian dialects. **At age of 15, he found errors in Laplace's work.** Hamilton's Principle of Least Action, and its associated equations and concept of configuration space, led to a revolution in mathematical physics. Hamilton also made revolutionary contributions to dynamics, differential equations, the theory of equations, numerical analysis, fluctuating functions, and graph theory.

Definition of Laplace transform

The Laplace transform is a method for solving differential equations. It has some advantages over the other methods, e.g.

- it will immediately give a particular solution satisfying given initial conditions,
- the driving function (function on the right side) can be discontinuous.



Like operator of differentiation converts function $f(x)$ into other function $f'(x)$, so the Laplace operator converts functions. By means of Laplace operator discontinuous function can be converted into other function.

The process of solving DE consists of 3 main steps:

1. Transform the given hard to solve problem into simple problem.
2. Solve the simple problem.
3. Transform the solution of simple problem back to have a solution of given problem.

The Laplace transform is closely related to Fourier transform. If Laplace transform is defined on interval from negative infinity to infinity (referred to as bilateral transform) then Fourier transform can be seen as a special case of Laplace transform.

Improper integral

Note: improper Riemann integral is such, that has a domain involving infinity (example: $\int_0^3 f(x)\,dx$ is proper,

$\int_0^\infty f(x)\,dx$ is improper integral).

Let $f(x)$ be a continuous function on the interval $I: 0 \le x \le h$. If, as $h \to \infty$, the definite integral $\int_0^h f(x)\,dx$ approaches a finite limit K, we say that improper integral $\int_0^\infty f(x)\,dx$ exists and **converges** to the value K (K is the area under the graph).

$$\int_0^\infty f(x)\,dx = \lim_{h\to\infty} \int_0^h f(x)\,dx = K$$

If the limit on the right does not exist, we say the improper integral on the left **diverges** and does not exist. Sometimes the result of improper integral is obvious, but sometimes we have to work with limits.

Example

Determine whether the integral $\int_0^\infty 1/(x+1)\,dx$ exists.

$$\int_0^h \frac{1}{x+1}\,dx = \log(x+1)|_{x=0}^h = \log(h+1)$$

As $h \to \infty,\ \log(h+1) \to \infty$. Hence the improper integral diverges and does not exist.

Example

Determine whether the integral $\int_0^\infty 1/(x^2+1)\,dx$ exists.

$$\int_0^h \frac{1}{x^2+1}\,dx = \arctan h - \arctan 0$$
$$\lim_{h\to\infty} (\arctan h) = \frac{\pi}{2}$$

Since $\arctan(0) = 0$ the integral exists and converges to $\pi/2$.

Laplace transform

If the improper integral

$$\int_0^\infty e^{-st} f(t)\,dt, \quad 0 \le t \le \infty, \tag{80}$$

converges for a value $s = s_0$, then it converges for every $s > s_0$. The integral, if exists, is called the Laplace transform of $f(t)$ and is written as $\mathscr{L}[f(t)]$. The variable t in (80) is substituted by limits and **the result of Laplace operator is** another **function of s,** e.g. $\mathscr{L}[f(t)] = F(s)$ or $\mathscr{L}[g(t)] = G(s)$.

An **operator** changes function $f(t)$ into $g(t)$. Here **transform** changes $f(t)$ into $F(s)$. The Laplace transform replaces power series: the power series (with n from zero to infinity) has to converge and converts given function into series. And in here we have improper integral for that task. The function e^t within integral is our favorite one: brings us convergence of the integral for positive s and is easy to integrate/differentiate.

Because we will often work with $\int e^{-st}\,dt$ it is useful to list following limits:

$$\lim_{t\to\infty} \frac{e^{-st}}{s} = 0, \quad \text{if } s > 0 \tag{81}$$
$$\lim_{t\to\infty} \frac{e^{-st}}{s^2} = 0, \quad \text{if } s > 0 \tag{82}$$
$$\lim_{t\to\infty} \frac{te^{-st}}{s} = 0, \quad \text{if } s > 0 \tag{83}$$
$$\lim_{t\to\infty} \frac{t^n e^{-st}}{s} = 0, \quad \text{if } s > 0,\ n \text{ real} \tag{84}$$

Note: function e^t is known to be one of the fastest growing function. Here is e^{-t} as one of the fastest decreasing function.

Example

Find the Laplace transform of the function $f(t) = 1$.

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st}\,dt = \lim_{h\to\infty}\int_0^h e^{-st}\,dt = \lim_{h\to\infty}\left.\frac{-e^{-st}}{s}\right|_{t=0}^{h} = \lim_{h\to\infty}\left(\frac{-e^{-st}}{s} + \frac{1}{s}\right)$$

$$\underline{\underline{\mathscr{L}[f(t)] = \frac{1}{s}}}$$

The result is valid for $s > 0$ (otherwise the integral does not converge—check numerator). The right branch of hyperbola is the Laplace transform of $f(t) = 1$.

Example

Find the Laplace transform of the function $f(t) = t$.

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st}t\,dt = \lim_{h\to\infty}\int_0^h e^{-st}t\,dt = \lim_{h\to\infty}\left.\frac{-e^{-st}t}{s} - \frac{e^{-st}}{s^2}\right|_{t=0}^{h} = (0-0) - (0 - \frac{e^0}{s^2})$$

$$\underline{\underline{\mathscr{L}[f(t)] = \frac{1}{s^2}}}$$

The result is valid for $s > 0$ (otherwise the integral does not converge) and the values from (82), (83) were used. The integral itself comes from integrating by parts, steps are not shown above.

Example

Find the Laplace transform of the function $f(t) = e^{-2t}$.

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st}e^{-2t}\,dt = \lim_{h\to\infty}\int_0^h e^{-(s+2)t}\,dt = \lim_{h\to\infty}\left.\frac{-e^{-(s+2)t}}{s+2}\right|_{t=0}^{h} = 0 - (\frac{-e^0}{s+2})$$

$$\underline{\underline{\mathscr{L}[f(t)] = \frac{1}{s+2}}}$$

The result is valid for $s > -2$ (otherwise the integral does not converge). From the second integral above you can see that the Laplace transform of e^{-2t} is again hyperbola, shifted to the left (exponential shift).

Example

Find the Laplace transform of the function $f(t) = e^{3t}$.

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st}e^{3t}\,dt = \lim_{h\to\infty}\int_0^h e^{-(s-3)t}\,dt = \lim_{h\to\infty}\left.\frac{-e^{-(s-3)t}}{s-3}\right|_{t=0}^{h} = 0 - (\frac{-e^0}{s-3})$$

$$\underline{\underline{\mathscr{L}[f(t)] = \frac{1}{s-3}}}$$

The result is valid for $s > 3$ (otherwise the integral does not converge).

Linearity of the Laplace transform

We like the fact, that if the Laplace transform of $f_1(t)$ converges for s_1 and the Laplace transform of $f_2(t)$ converges for s_2 then for s greater than both s_1, s_2

$$\begin{gathered}\mathscr{L}[f_1 + f_2] = \mathscr{L}[f_1] + \mathscr{L}[f_2],\\ \mathscr{L}[cf] = c\mathscr{L}[f],\\ \mathscr{L}[c_1 f_1 + c_2 f_2] = c_1\mathscr{L}[f_1] + c_2\mathscr{L}[f_2],\end{gathered}$$

where $c_1,\ c_2$ are constants, i.e. the Laplace transform is a linear operator.

Example: for $s > 0$: $\quad \mathscr{L}[1 + 5t] = \mathscr{L}[1] + 5\mathscr{L}[t] = 1/s + 5/s^2$.

Example: for $s > 3$: $\quad \mathscr{L}[4e^{3t} - 5t] = 4\mathscr{L}[e^{3t}] - 5\mathscr{L}[t] = 1/(s-3) - 5/s^2$.

Transforms of some basic functions

(1) $\mathscr{L}[1] = \dfrac{1}{s}$	**(2)** $\mathscr{L}[t^n] = \dfrac{n!}{s^{n+1}}\ ,\ n = 1,\ 2,\ 3,\ \ldots$
(3) $\mathscr{L}[e^{at}] = \dfrac{1}{s-a}$	**(4)** $\mathscr{L}[e^{at}t^n] = \dfrac{n!}{(s-a)^{n+1}}\ ,\ n = 0,\ 1,\ \ldots$
(5) $\mathscr{L}[\sin kt] = \dfrac{k}{s^2 + k^2}$	**(6)** $\mathscr{L}[\cos kt] = \dfrac{s}{s^2 + k^2}$
(7) $\mathscr{L}[\sinh kt] = \dfrac{k}{s^2 - k^2}$	**(8)** $\mathscr{L}[\cosh kt] = \dfrac{s}{s^2 - k^2}$

Sufficient conditions for existence of Laplace transform

Some functions do not posses Laplace transform. Transform $\mathscr{L}[e^{t^2}]$ does not exist as integral $\int_0^\infty e^{-st}e^{t^2}\,dt$ does not exist (e^{t^2} grows much faster than e^{-st} decreases); similarly $\mathscr{L}[1/t]$ does not exist as the integral (80) diverges (e.g. on the interval $(0,1)$).

To the definition (80) we add sufficient conditions for existence of $\mathscr{L}[f(t)]$:

- **Function $f(t)$ have to be piecewise continuous on $[0, \infty\rangle$**
- **Function $f(t)$ is of exponential order for some T such, that $t > T$**
  There exist constants $c,\ M > 0$ such that for every $t > T$: $|f(t)| < Me^{ct}$. In other words: from a point T the function $f(t)$ grows slower than function Me^{ct}.

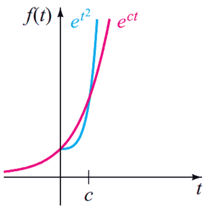


FIGURE 7.1.4 e^{t^2} is not of exponential order

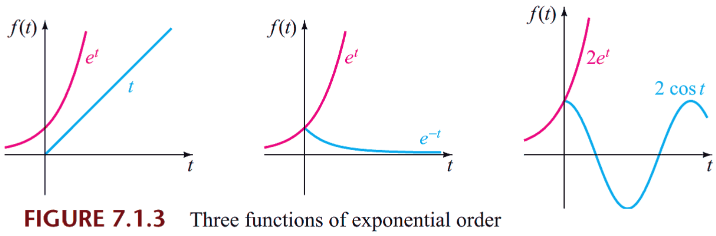


FIGURE 7.1.3 Three functions of exponential order

These conditions are not necessary. E.g. $f(t) = 1/\sqrt{t}$ is not piecewise continuous but its transform does exist.

Transform of piecewise continuous functions

Example

Find the Laplace transform of the function

$$f(t) = \begin{cases} 1-t, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ e^{2t}, & 3 < t < \infty \end{cases}$$

$$
\begin{aligned}
\mathscr{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}(1-t)\,dt + \int_1^3 e^{-st}\cdot 0\,dt + \int_3^\infty e^{-st}e^{2t}\,dt = \\
&= \int_0^1 e^{-st}\,dt - \int_0^1 te^{-st}\,dt + \int_3^\infty e^{(2-s)t}\,dt = \\
&= -\frac{1}{s}e^{-st}\Big|_{t=0}^{1} - \frac{-(st+1)e^{-st}}{s^2}\Big|_0^1 + \frac{e^{(2-s)t}}{2-s}\Big|_3^\infty = \\
&= -\frac{1}{s}e^{-st}\Big|_0^1 - \frac{-(st+1)e^{-st}}{s^2}\Big|_0^1 + \frac{e^{(2-s)t}}{2-s}\Big|_3^\infty = \\
&= \left(-\frac{1}{s}e^{-s} - (-\frac{1}{s})\right) - \left(\frac{-(s+1)e^{-s}}{s^2} - \frac{-1}{s^2}\right) + \left(0 - \frac{e^{(2-s)\cdot 3}}{2-s}\right) = \\
&= \underline{\underline{\frac{e^{-s}+s-1}{s^2} + \frac{e^{6-3s}}{s-2}}}
\end{aligned}
$$

For $s > 2$. Note that the integral $\int_3^\infty e^{(2-s)t}\,dt$ does not exist if $s \le 2$. Note also that the result is not piecewise defined.

Inverse transforms

If $F(s)$ represents the Laplace transform of $f(t)$, i.e. $\mathscr{L}[f(t)] = F(s)$, then $f(t)$ is the inverse Laplace transform of $F(s)$.

$$f(t) = \mathscr{L}^{-1}[F(s)] \iff F(s) = \mathscr{L}[f(t)]$$

Inverse transform is the hardest part of computing DE by Laplace transform. Finally the function $F(s)$ has to be converted back to $f(t)$. For such task, we can not use tables unless they would be too long to be useful. The work mostly consists of partial fractions decomposition.

Inverse transforms of some basic functions

(1) $1 = \mathscr{L}^{-1}\left[\frac{1}{s}\right]$	**(2)** $t^n = \mathscr{L}^{-1}\left[\frac{n!}{s^{n+1}}\right],\ n = 1,\ 2,\ 3,\ \ldots$
(3) $e^{at} = \mathscr{L}^{-1}\left[\frac{1}{s-a}\right]$	**(4)** $e^{at}t^n = \mathscr{L}^{-1}\left[\frac{n!}{(s-a)^{n+1}}\right],\ n = 0,\ 1,\ \ldots$
(5) $\sin kt = \mathscr{L}^{-1}\left[\frac{k}{s^2+k^2}\right]$	**(6)** $\cos kt = \mathscr{L}^{-1}\left[\frac{s}{s^2+k^2}\right]$
(7) $\sinh kt = \mathscr{L}^{-1}\left[\frac{k}{s^2-k^2}\right]$	**(8)** $\cosh kt = \mathscr{L}^{-1}\left[\frac{s}{s^2-k^2}\right]$

Example

Find $\mathscr{L}^{-1}\left[1/(\mathrm{s}-1)\right],\ \mathrm{s} > 1$

From the above table we know that

$$e^{at} = \mathscr{L}^{-1}\left[\frac{1}{s-a}\right]$$

We find that for $a = 1$ $e^t = \mathscr{L}^{-1}[1/s-1]$. Hence $\mathscr{L}^{-1}[1/s-1] = e^t$ for $t > 0$.

Example

Find $\mathscr{L}^{-1}\left[(5\mathrm{s}-2)/(\mathrm{s}^2+4)\right]$.

$$\mathscr{L}^{-1}\left[\frac{5\mathrm{s}-2}{\mathrm{s}^2+4}\right]=\mathscr{L}^{-1}\left[\frac{5\mathrm{s}}{\mathrm{s}^2+4}\right]-\mathscr{L}^{-1}\left[\frac{2}{\mathrm{s}^2+4}\right]=5\cos 2\mathrm{t}-\sin 2\mathrm{t}.$$

From the table of transforms:

$$\mathbf{(5)}\ \sin kt=\mathscr{L}^{-1}\left[\frac{k}{s^2+k^2}\right],\qquad \mathbf{(6)}\ \cos kt=\mathscr{L}^{-1}\left[\frac{s}{s^2+k^2}\right]$$

The match for the member $\mathscr{L}^{-1}\left[5\mathrm{s}/(\mathrm{s}^2+4)\right]$ is $\cos kt$. Then $k=2$.

$$k=2:\ \mathscr{L}[\cos 2t]=\frac{s}{s^2+4}$$

The match for the member $\mathscr{L}^{-1}\left[2/(\mathrm{s}^2+4)\right]$ is $\sin kt$. Then $k=2$.

$$k=2:\ \mathscr{L}[\sin 2t]=\frac{2}{s^2+4}$$

Decomposition to partial fractions

The differential equation is solved in the domain s where it is easy to solve. Finaly inverse Laplace transform on solution $Y(s)=p(s)/q(s)$, where $p(s),\ q(s)$ are **polynomials**, has to be applied. The polynomial $Y(s)$ is usually too complicated for inverse transform and has to be splitted into simple fraction according to rules shown below.

Case	**Polynomial observed**	**Rule for decomposition**
Real roots	$(s-a)$	$\frac{A}{s-a}$
Repeated roots	$(s-a)^2$	$\frac{A}{s-a}+\frac{A}{(s-a)^2}$
Imaginary roots	(s^2+a^2)	$\frac{As+B}{s^2+a^2}$

Example: Case I (real roots)

Find $\mathscr{L}^{-1}[F],\ F=(6s^2+5s-3)/(s^3+2s^2-3s)$.

$F(s)=(6s^2+5s-3)/(s^3+2s^2-3s)$ can be rewritten into $F(s)=\frac{1}{s}+\frac{3}{s+3}+\frac{2}{s-1}$.

To convert $(6s^2+5s-3)/(s^3+2s^2-3s)$ into $\frac{1}{s}+\frac{3}{s+3}+\frac{2}{s-1}$ we have to employ partial fraction decomposition:

$$\frac{6s^2+5s-3}{s^3+2s^2-3s}=\frac{6s^2+5s-3}{s(s+3)(s-1)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-1}$$

By multipliyng both sides by $s(s+3)(s-1)$ we have

$$\begin{aligned}6s^2+5s-3=&A(s+3)(s-1)+Bs(s-1)+Cs(s+3)\implies\\ \implies 6s^2+5s-3=&As^2+2As-3A+Bs^2-Bs+Cs^2+3Cs\\ 6s^2+5s-3=&\ (A+B+C)s^2+(2A-B+3C)s+1(-3A)\cdot 1\implies\\ \implies A=&1,\ B=3,\ C=2\end{aligned}$$

We used the method of undetermined coefficients and here comes

$$\frac{6s^2+5s-3}{s^3+2s^2-3s}=\frac{1}{s}+\frac{3}{s+3}+\frac{2}{s-1}.$$

Then

$$\mathscr{L}^{-1}\left[\frac{1}{s}+\frac{3}{s+3}+\frac{2}{s-1}\right]=\mathscr{L}^{-1}\left[\frac{1}{s}\right]+\mathscr{L}^{-1}\left[\frac{3}{s+3}\right]+\mathscr{L}^{-1}\left[\frac{2}{s-1}\right]$$

$$\mathscr{L}^{-1}\left[\frac{1}{s}\right]=1$$

$$\mathscr{L}^{-1}\left[\frac{3}{s+3}\right]=3e^{-3t}$$

$$\mathscr{L}^{-1}\left[\frac{2}{s-1}\right]=2e^{t}$$

And finally

$$\underline{\underline{f(t)=1+3e^{-3t}+2e^{t}}}.$$

Example: Case II (repeated roots)

Find $\mathscr{L}^{-1}[F],\ F=s^2/(s+1)^3$.

$F(s)=s^2/(s+1)^3$ can be rewritten into $F(s)=\frac{1}{(s+1)}-\frac{2}{(s+1)^2}+\frac{1}{(s+1)^3}$.

To convert $s^2/(s+1)^3$ into $\frac{1}{(s+1)}-\frac{2}{(s+1)^2}+\frac{1}{(s+1)^3}$ we have to employ partial fraction decomposition:

$$\frac{s^2}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}$$

By multipliyng both sides by $(s+1)^3$ we have

$$\begin{aligned} s^2 =& A(s+1)^2+B(s+1)+C \implies \\ \implies s^2 =& As^2+2As+A+Bs+B+C \\ s^2 =& As^2+(2A+B)s+(A+B+C) \implies \\ \implies A =& 1,\ B=-2,\ C=1 \end{aligned}$$

We used the method of undetermined coefficients and here comes that

$$\frac{s^2}{(s+1)^3}=\frac{1}{s+1}-\frac{2}{(s+1)^2}+\frac{1}{(s+1)^3}$$

Then

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{s+1}-\frac{2}{(s+1)^2}+\frac{1}{(s+1)^3}\right] =& \mathscr{L}^{-1}\left[\frac{1}{s+1}\right]-\mathscr{L}^{-1}\left[\frac{2}{(s+1)^2}\right]+ \\ & +\mathscr{L}^{-1}\left[\frac{1}{(s+1)^3}\right] \\ \mathscr{L}^{-1}\left[\frac{1}{s+1}\right] =& e^{-t} \\ \mathscr{L}^{-1}\left[\frac{2}{(s+1)^2}\right] =& 2e^{-t}t \\ \mathscr{L}^{-1}\left[\frac{1}{(s+1)^3}\right] =& \frac{1}{2}e^{-t}t^2 \end{aligned}$$

And finally

$$\underline{\underline{f(t)=e^{-t}-2e^{-t}t+\frac{1}{2}e^{-t}t^2}}.$$

Example: Case III (imaginary quadratic roots)

Find $\mathscr{L}^{-1}[F],\ F = (1+\pi s)/[s^2(s^2+1)]$.

$F = (1+\pi s)/[s^2(s^2+1)]$ can be rewritten into $\frac{\pi}{s} + \frac{1}{s^2} - \frac{\pi s+1}{s^2+1}$.

To convert $(1+\pi s)/[s^2(s^2+1)]$ into $\frac{\pi}{s} + \frac{1}{s^2} - \frac{\pi s+1}{s^2+1}$ we can employ partial fraction decomposition:

$$\frac{1+\pi s}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \tag{85}$$

By multiplying both sides by $s^2(s^2+1)$ we get

$$\begin{aligned} 1+\pi s &= As(s^2+1) + B(s^2+1) + Cs(s^2) + Ds^2 \implies \\ \implies 1+\pi s &= As^3 + As + Bs^2 + B + Cs^3 + Ds^2 \\ 1+\pi s &= s^3(A+C) + s^2(B+D) + s(A) + 1(B) \implies \\ &\implies A=\pi,\ B=1,\ C=-\pi,\ D=-1 \end{aligned}$$

Now we can place the found constants back to (85):

$$\frac{1+\pi s}{s^2(s^2+1)} = \frac{\pi}{s} + \frac{1}{s^2} + \frac{-\pi s-1}{s^2+1}$$

That suggests we have to find inverse transform of below terms in order to collect $\mathscr{L}^{-1}[F]$.

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{\pi}{s}\right] &= \pi \\ \mathscr{L}^{-1}\left[\frac{1}{s^2}\right] &= t \\ \mathscr{L}^{-1}\left[\pi\frac{s}{s^2+1}\right] &= \pi\cos t \\ \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] &= \sin t \end{aligned}$$

The solution we are looking for is then

$$\underline{\underline{y(t) = \pi + t - \pi\cos t - \sin t.}}$$

Laplace transform

Solving differential equation by the Laplace transform

We are going to solve differential equation in the form

$$\begin{aligned} y'' + a_1 y' + a_0 y &= g(t) \\ y(0) &= y_0, \\ y'(0) &= y_0'. \end{aligned}$$

First, some important observations of above:

- The Laplace transform is intended for solving linear DE: **linear DE are transformed into algebraic ones.** If the given problem is nonlinear, it has to be converted into linear. Or other method have to be used instead (e.g. numerical method).
- The Laplace solves DE from time $t = 0$ to infinity. I.e. **initial values** at $t = 0$, where the problem starts, **are part of the problem** and have to be given. If there are no initial values, we have to work with generic constants/symbols $(y_0,\ y_0')$.
- The strength of the method is that **function $g(x)$ on the right side can be piecewise defined, periodic or impulsive.**

How the method works

We are looking for $y(t)$ as a solution of differential equation. Because of convention used, such function is $Y(s)$ when transformed by Laplace.

1. **Laplace transform**
   This step is relatively easy.
2. **Algebraic equation** in $Y(s)$
   That means equation is being solved in the domain of $Y(s)$, where it is easy to solve. Result is $Y(s) = p(s)/q(s)$ where $p(s),\ q(s)$ are polynomials. $Y(s)$ is the Laplace transform of solution.
3. **Inverse transform**
   Inverse Laplace transform is the hardest part.

If we want to apply the Laplace method, we have to be able to transform each member of DE. That means, we have to know also how to deal with $y',\ y''$.

$$\mathscr{L}\left[f'(t)\right] = \int_0^\infty e^{-st} f'(t)\, dt \tag{86}$$

So let us integrate. There are two functions multiplied, therefore we have to integrate by parts (use the product rule backwards). When using product rule, we want $f'(t)$ to integrate (there would be no outcome if we choose $f'(t)$ to differentiate, you can try yourself), so e^{-st} has to be differentiated.

$$\begin{aligned} (uv)' = u'v + uv' \implies u'v = (uv)' - uv' \implies \\ \int u'v = uv - \int uv' \\ v' = f'(t), \qquad u = e^{-st} \end{aligned} \tag{87}$$

Product rule has been reviewed and u, v have been identified. Now we can enter u, v' into (87) in order to resolve (86):

$$\begin{aligned} \mathscr{L}\left[f'(t)\right] = e^{-st} f(t)\big|_{t=0}^{\infty} - \int_{t=0}^{\infty} -se^{-st} f(t)\, dt = (0 - f(0)) + sF(s) \implies \\ \implies \mathscr{L}\left[f'(t)\right] = sF(s) - f(0) \end{aligned} \tag{88}$$

The Riemann integral has been evaluated with boundaries for t, so the variable t has served to find Laplace transform $F(s)$ of $f'(t)$.

Since we are going to solve 2nd order DE, it is essential to find also Laplace transform of $f''(t)$. The equation (88) can be written as $\mathscr{L}\left[f'(t)\right] = s\mathscr{L}[f(t)] - f(0)$. And then what works for $f'(t)$ works also for $f''(t)$:

$$\mathscr{L}\left[f''(t)\right] = s\mathscr{L}[f'(t)] - f'(0) = s\left[sF(s) - f(0)\right] - f'(0) \implies$$
$$\implies \mathscr{L}\left[f''(t)\right] = s^2F(s) - sf(0) - f'(0) \tag{89}$$

These important equations (88) and (89) can be sumarized into:

$$\begin{aligned} \mathscr{L}\left[y\right] &= Y \\ \mathscr{L}\left[y'\right] &= sY - y(0) \\ \mathscr{L}\left[y''\right] &= s^2Y - sy(0) - y'(0) \end{aligned}$$

Example

Show the Laplace transform of DE $y'' - y = e^{-t},\ y(0) = 1,\ y'(0) = 0$.

$$s^2Y - s - Y = \frac{1}{s+1}$$

Note: $Y = Y(s)$ is Laplace transform of $y(t)$.

Shifting theorems

There are two shifting theorems to deal with.

1. The **first shifting theorem** ($f(t)$ is multiplied by e^{at})
   If $f(t)$ is multiplied by e^{at}, then $\mathscr{L}[f(t)e^{at}] = F(s-a)$.
2. The **second shifting theorem** talks about shifting the t axis.
   A. We want to shift the function to $t = a$. If $F(s) = \mathscr{L}[f(t)]$ then

   $$u_a(t)\ f(t-a) = e^{-as}F(s).$$

   B. We want to erase part of the function. Then

   $$u_a(t)\ f(t) = e^{-as}\mathscr{L}[f(t+a)].$$

   The function $u_a(t)$ above is a unit step function at $t = a$ and the shifting theorem is useful when working with **piecewise continuous functions.**

The first shifting theorem

Sometimes the given function $f(t)$ might be multiplied by e^{at}. If $F(s)$ is known, then we can evaluate $\mathscr{L}[e^{et}f(t)]$ without integrating, which is very helpful. See also the example from the previous chapter.

$$\mathscr{L}[e^{at} \cdot f(t)] = F(s-a) \qquad \text{if} \qquad \mathscr{L}[f(t)] = F(s).$$

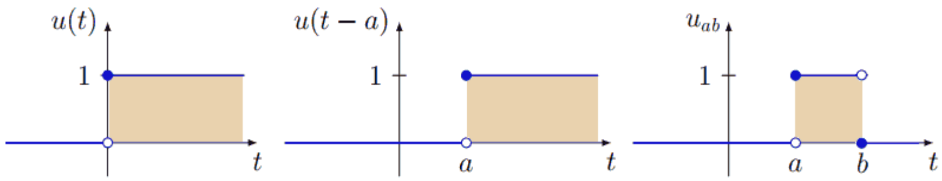
Example: $\mathscr{L}[1] = 1/s \implies \mathscr{L}[e^{at}] = 1/(s-a)$.

Example: $\mathscr{L}[\cos kt] = s/(s^2 + k^2) \implies \mathscr{L}[\cos kt \cdot e^{at}] = ((s-a)/(s-a)^2 + k^2)$.

Piecewise continuous functions

One of the important property of Laplace transform is that we can wipe out the value of a function outside an interval ab.

For that reason we define **unit step function** and **unit box function.**



Unit step function $u(t)$, $u(t-a)$ and unit box function u_{ab}. Please realize, that if we place unit step function u_a into a and add other unit step function $-1 \cdot u_b$ into b then the sum of these graphs will be depicted as a box function u_{ab}.

Unit step function $u(t)$ in its basic form is defined as either 0 or 1 (hence *unit*):

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < \infty \end{cases} \tag{90}$$

So the value of that function is either zero or one, depends whether we look to the left or the right from the vertical axis. Any function $f(t)$ can be multiplied by $u(t)$. Then the left branch of the graph will be wiped out while the right branch is left untouched.

Things usually go more complicated and one might need the step not in zero. Then unit step function is $u(t-a)$ (or u_a). Again, we usually need such function to multiply other function $f(t)$.

Unit box function is a combination of two unit step functions. Let us say first we have positive jump in a and then a negative jump in b.

$$u_{ab} = u_a(t) - u_b(t) = u(t-a) - u(t-b)$$

The second shifting theorem

Now arrives the need to operate with unit step functions. These unit functions are somehow non-standard so we need some rules for handling them properly. If we want to shift the t axis of a function, then we move the function to a. Then instead of t we have $(t-a)$ (what was at zero, now is at a). We have also to multiply such function by $u(t-a)$ to erase whatever would be considered before $t=0$. Remember that Laplace starts at time is zero and the past can not be recovered, so it has to be set to zero. Then

$$\mathscr{L}[u_a(t)f(t-a)] = e^{-as}F(s). \tag{91}$$

To erase branch of the given function until $t = a$: we multiply the function by unit step function $u_a(t)$ and evaluate its transform by the rule

$$\mathscr{L}[u_a(t)f(t)] = e^{-as}\mathscr{L}[f(t+a)]. \tag{92}$$

The member e^{-as} "remembers" the shift of t axis. The rules (91) and (92) can be derived from the integral–definition of Laplace transform.

Example

Show the Laplace transform of unit box u_{ab}

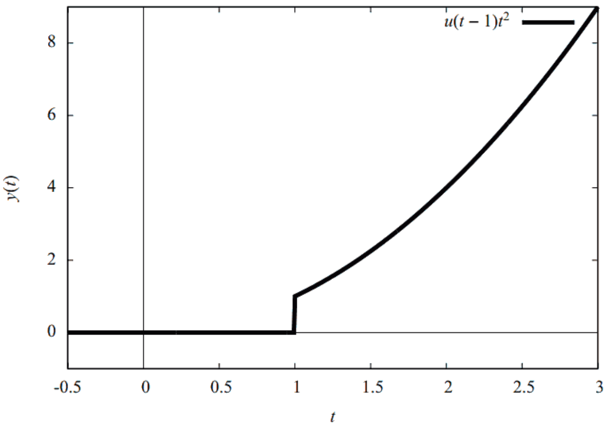
$$u_{ab} = u(t-a) - u(t-b)$$
$$\mathscr{L}\left[u(t-a)\right] = e^{-as}\,\frac{1}{s}$$
$$\mathscr{L}\left[u(t-b)\right] = e^{-bs}\,\frac{1}{s}$$
$$\underline{\underline{\mathscr{L}\left[u_{ab}\right] = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}}}$$

Example

Show the Laplace transform of t^2 when the function until $t = 1$ is required to be erased to zero.

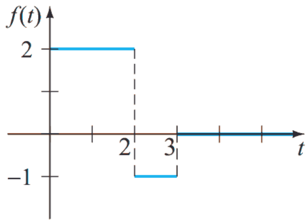
We have to erase whatever would be before $t=1$, i.e. the function $f(t)$ have to be multiplied by unit step function with the jump from 0 to 1 at $a=1$: $u_a(t) = u(t-1)$. In order to multiply we have to employ the rule (92) for multiplication of the function with function of the type unit step.

$$\mathscr{L}\left[u(t-1)\,t^2\right] = e^{-as}\mathscr{L}\left[(t+1)^2\right] = e^{-as}\mathscr{L}\left[t^2+2t+1\right] = \underline{\underline{e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)}}$$



Example

Describe function depicted on graph by unit step functions and find its Laplace transform.



1. At $t = 0$ the function looks like the very basic unit step function (90). But unit function knows only about 0 and 1, here we have $f(t) = 2$. That means we have to use $2u(t)$.
2. Then in time $t = 2$ its value changes from 2 to -1 (i.e. 3 **down** at $t = 2$) which means we have to add $-3u(t-2)$.
3. Finally the value at $t = 3$ jumps 1 higher, which brings member $u(t-3)$.

$$f(t) = 2u(t) - 3u(t-2) + u(t-3)$$

So far we collected unit step functions to express function from the graph. Now we have to use the rule (91) to evaluate their Laplace transform.

$$\mathscr{L}[f(t)] = \mathscr{L}[2u(t)] - \mathscr{L}[3u(t-2)] + \mathscr{L}[u(t-3)] = \\ = \underline{\underline{\frac{2}{s} - 3e^{-2s}\frac{1}{s} + e^{-3s}\frac{1}{s}}}$$

Example

Find $f(t)$ if $F(s) = e^{-2s}/(s^2+1)$.

We are given result in the domain s and we have to use inverse Laplace transform. **Since member e^{-2s} is present, it is known that either a shift or intervals are involved.** In this case it can be identified that the shift was made to $a = 2$. The other member $1/(s^2+1)$ is—according to the table of transforms—the product of $\sin kt$ when $k = 1$.

So we have

$$\underline{\underline{f(t) = u(t-2)\sin(t-2)}} \qquad \text{or} \qquad \underline{\underline{f(t) = u_2(t)\sin(t-2)}}.$$

It is $\sin()$ moved on t axis from zero to 1 and in such case the unit step function has to be involved to erase whatever was before the beginning.

Example

Find $f(t)$ if

$$F(s) = \frac{1+e^{-\pi s}}{s^2+1}$$

We are given result in the domain s and we have to use inverse Laplace transform. **Since member $e^{-\pi s}$ is present, it is known that either a shift or interval is involved.** In this case it can be identified that the shift was made to $a = \pi$. Now let us split the fraction into two and solve them separately.

$$F(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

It comes from observation and from the table of transform that the first fraction is a product of $\sin kt$ when $k = 1$.

The second fraction involves shift to $a = \pi$ and $\sin()$ as well. You will find that the second fraction corresponds to $f(t) = u(t-\pi)\sin(t-\pi)$.

So we have piecewise function, let us combine both members together.

$$f(t) = \sin t + u_\pi(t)\sin(t-\pi)$$

What we see is that the first part $(\sin t)$ is valid for the whole interval, while the second part is erased by u_π for $t < \pi$ and for $t \geq \pi :\ \sin(t-\pi)$ comes into life.

We have enough information to transform $f(t)$ into the *case form*:

$$f(t) = \begin{cases} \sin t & 0 \leq t < \pi \\ \sin t + 1 \cdot \sin(t-\pi) & t \geq \pi \end{cases}$$

The second case deserves a note: it starts with $\sin t$ because this member is valid from $t > 0$. The number 1 is value of $u_\pi(t)$ when $t \geq \pi$, because that is the purpose of the unit step function (to be one or zero). Finally $\sin(t-\pi)$ is taken.

Now you can read from graph of $\sin()$ that $\sin(t) = -\sin(t-\pi)$. The consequence is that in the second case both sines cancel each other:

$$f(t) = \begin{cases} \sin t & 0 \leq t < \pi \\ 0 & t \geq \pi \end{cases}$$

And that is the final answer.

Example

Solve DE $y'' + y = t$ with initial values $y(\pi) = 0,\ y'(\pi) = 0$

There is an inconvenience that the inital values are described at $t = \pi$ but we expect them at $t = 0$. We will convert DE into other DE which has origin at π:

$$w'' + w = t + \pi, \qquad w(0) = 0,\ w'(0) = 0 \tag{93}$$

We have moved with the origin of t axis to π, so the nonhomogenous term of new DE (93) has to have value of $t + \pi$. You can check that now at $t = 0$ within w everything behaves like at $t = \pi$ within y. We used **substitution $w(t) = y(t+\pi)$.** Now the DE is ready to be transformed and solved:

$$\begin{aligned} (s^2W - s\cdot 0 - 0) + (s \cdot 0 - 0) + W &= \frac{1}{s^2} + \frac{\pi}{s} \\ W(s^2+1) &= \frac{1}{s^2} + \frac{\pi}{s} \\ W &= \left(\frac{1}{s^2} + \frac{\pi}{s}\right)\frac{1}{s^2+1} \\ W &= \frac{1+\pi s}{s^2(s^2+1)} \end{aligned}$$

We have Laplace transform of solution $w(t)$. The inverse transform of $W(s)$ is made in example solved before.

$$w(t) = \pi + t - \pi\cos t - \sin t$$
$$\underline{\underline{y(t)}} = \pi + (t-\pi) - \pi\cos(t-\pi) - \sin(t-\pi) = \underline{\underline{t + \pi\cos t + \sin t}}$$

Example

Solve DE $y'' - 5y' + 6y = e^t(2t-3)$ with initial values $y(0) = 1,\ y'(0) = -1$

We have to transform each term to obtain equation in domain of $Y(s)$, where it is easy to solve. We will use (88), (89) and table of transforms of basic functions (for the nonhomogenous members on the right side of DE).

$$s^2Y - s - (-1) - 5(sY - 1) + 6Y = 2\frac{1}{(s-1)^2} - 3\frac{1}{s-1}$$
$$s^2Y - s + 1 - 5sY + 5 + 6Y = 2\frac{1}{(s-1)^2} - 3\frac{1}{s-1}$$
$$Y(s^2 - 5s + 6) = 2\frac{1}{(s-1)^2} - 3\frac{1}{s-1} - 6 + s$$
$$Y = \frac{2 - 3(s-1) - 6(s-1)^2 + s(s-1)^2}{(s-1)^2} \cdot \frac{1}{s^2 - 5s + 6}$$
$$Y = \frac{2 - 3s + 3 - 6(s^2 - 2s + 1) + s(s^2 - 2s + 1)}{(s-1)^2(s-2)(s-3)}$$
$$Y = \frac{5 - 3s - 6s^2 + 12s - 6 + s^3 - 2s^2 + s}{(s-1)^2(s-2)(s-3)}$$
$$Y = \frac{-1 + 10s - 8s^2 + s^3}{(s-1)^2(s-2)(s-3)}$$

That was basic algebra in s domain. We have to continue with decomposition into partial fractions.

$$\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2} + \frac{D}{s-3} = \frac{-1 + 10s - 8s^2 + s^3}{(s-1)^2(s-2)(s-3)}$$
$$A(s-1)(s-2)(s-3) + B(s-2)(s-3) + \\ +C(s-3)(s-1)^2 + D(s-2)(s-1)^2 = -1 + 10s - 8s^2 + s^3$$
$$A(s^3 - 6s^2 + 11s - 6) + B(s^2 - 5s + 6) + \\ +C(s^3 - 5s^2 + 7s - 3) + D(s^3 - 4s^2 + 5s - 2) = -1 + 10s - 8s^2 + s^3$$
$$s^3(A + C + D) + s^2(-6A + B - 5C - 4D) + \\ +s(11A - 5B + 7C + 5D) + (-6A + 6B - 3C - 2D) = -1 + 10s - 8s^2 + s^3$$

When system of 4 linear equations is solved we have three fractions, each of them represents one term of solution of DE.

$$\left.\begin{aligned} A + C + D &= 1 \\ -6A + B - 5C - 4D &= -8 \\ 11A - 5B + 7C + 5D &= 10 \\ -6A + 6B - 3C - 2D &= -1 \end{aligned}\right\} \qquad \begin{aligned} A &= 0 \\ B &= 1 \\ C &= 5 \\ D &= -4 \end{aligned}$$
$$Y(s) = \frac{1}{(s-1)^2} + \frac{5}{s-2} - \frac{4}{s-3}$$
$$\underline{\underline{y(t) = e^{-t}t + 5e^{-2t} - 4e^{-3t}}}$$

Example

Solve DE $y'' + y' = \pi$ with initial values $y(0) = \pi,\ y'(0) = 0$

We have to transform each term to obtain equation in domain of $Y(s)$, where it is easy to solve. We will use (88), (89) and table of transforms of basic functions (for the nonhomogenous members on the right side of DE).

$$\begin{aligned}(s^2Y - s\pi - 0) + (sY - \pi) &= \pi\frac{1}{s}\\ s^2Y - s\pi + sY - \pi &= \frac{\pi}{s}\\ Y(s^2+s) &= \frac{\pi}{s} + \pi + s\pi\\ Y &= \frac{\pi + \pi s + s^2\pi}{s(s^2+s)}\\ Y &= \frac{\pi(1+s+s^2)}{s\cdot s(s+1)}\end{aligned}$$

The fraction is too difficult for inverse transform. We have to split it into pieces by means of decomposition to partial fractions.

$$\begin{aligned}\frac{\pi(1+s+s^2)}{s\cdot s(s+1)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}\\ \pi(1+s+s^2) &= A(s)(s+1) + B(s+1) + Cs^2\\ \pi + \pi s + \pi s^2 &= As^2 + As + Bs + B + Cs^2\\ \pi + \pi s + \pi s^2 &= s^2(A+C) + s(A+B) + 1(B) \implies\\ \implies\ A = 0,\ B = \pi,\ C = \pi\end{aligned}$$

The solution is splitted into simple fractions:

$$Y(s) = \frac{\pi}{s^2} + \frac{\pi}{s+1}$$

The first fraction is Laplace transform of πt, the second fraction can be identified as a Laplace transform of πe^{-t}.

$$\underline{\underline{y(t) = \pi t + \pi e^{-t}}}$$

Example

Solve DE $y'' + y' = e^t$ with initial values $y(0) = 1,\ y'(0) = 0$

We have to transform each term to obtain equation in domain of $Y(s)$, where it is easy to solve. We will use (88), (89) and table of transforms of basic functions (for the nonhomogenous members on the right side of DE).

$$\begin{aligned}(s^2Y - s - 0) + (sY - 1) &= \frac{1}{s-1}\\ s^2Y - s + sY - 1 &= \frac{1}{s-1}\\ Y(s^2+s) &= \frac{1}{s-1} + 1 + s\\ Y &= \frac{1 + (s-1) + s(s-1)}{(s-1)(s^2+s)}\\ Y &= \frac{1 + s - 1 + s^2 - s}{(s-1)s(1+s)}\end{aligned}$$

The fraction is too difficult for inverse transform. We have to split it into pieces by means of decomposition to partial fractions.

$$\begin{aligned}\frac{s^2}{(s-1)s(1+s)} &= \frac{A}{s-1}+\frac{B}{s}+\frac{C}{s+1}\\ s^2 &= A(s)(s+1)+B(s-1)(s+1)+C(s-1)(s)\\ s^2 &= As^2+As+Bs+Bs^2-B+Cs^2-Cs\\ s^2 &= s^2(A+B+C)+s(A-C)-1(B) \implies\\ \implies A &= 1/2,\ B=0,\ C=1/2\end{aligned}$$

The solution is splitted into simple fractions:

$$Y(s)=\frac{1}{2}\cdot\frac{1}{s-1}+\frac{1}{2}\cdot\frac{1}{s+1}$$

The first fraction is Laplace transform of e^{-t}, the second fraction can be identified as a Laplace transform of e^t.

$$\underline{\underline{y(t)=\frac{1}{2}e^{-t}+\frac{1}{2}e^t}}$$

Example

Solve DE $y''-y=\sin t$ with initial values $y(0)=-1,\ y'(0)=0$

We have to transform each term to obtain equation in domain of $Y(s)$, where it is easy to solve. We will use (88), (89) and table of transforms of basic functions (for the nonhomogenous members on the right side of DE).

$$\begin{aligned}(s^2Y+s-0)-(Y) &= \frac{1}{s^2+1}\\ s^2Y-Y &= \frac{1}{s^2+1}-s\\ Y(s^2-1) &= \frac{1-s^3-s}{s^2+1}\\ Y &= \frac{1-s^3-s}{(s^2+1)(s-1)(s+1)}\end{aligned}$$

Such fraction is too difficult for inverse transform. We have to split it into pieces by means of decomposition to partial fractions.

$$\frac{1-s^3-s}{(s^2+1)(s-1)(s+1)}=\frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{s+1}$$

Let us multiply both sides by $(s^2+1)(s-1)(s+1)$:

$$\begin{aligned}1-s^3-s &= (As+B)(s-1)(s+1)+C(s^2+1)(s+1)+D(s^2+1)(s-1)\\ 1-s^3-s &= As^3+Bs^2-As-B+Cs^3+Cs^2+Cs+C+Ds^3-Ds2+Ds-D\\ 1-s^3-s &= s^3(A+C+D)+s^2(B+C-D)+s(-A+C+D)+1(-B+C-D)\end{aligned}$$

We are employing method of undetermined coefficients to find values A, B, C, D. That means there is a system of linear equations (the first one is $-1=A+C+D$) which is not shown and solved in here. Once the system is solved, the results are

$$\begin{aligned}A=0,\ B=-\frac{1}{2},\ C=-\frac{1}{4},\ D=-\frac{3}{4} \implies\\ \implies Y(s)=-\frac{1}{2}\cdot\frac{1}{s^2+1}-\frac{1}{4}\cdot\frac{1}{s-1}-\frac{3}{4}\,\frac{1}{s+1}\end{aligned}$$

We can use table of transforms of basic function to identify that the first fraction is transform of $\sin t$, the second e^{-t} and the last one is e^t.

$$\underline{\underline{y(t)=-\frac{1}{2}\sin t-\frac{1}{4}e^{-t}-\frac{3}{4}e^t}}$$

Example

Solve the system of DE

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + 2y$$

No initial values are given, we have to provide arbitrary values $x(0) = x_0,\ y(0) = y_0$. First we transform both equations by Laplace and then we solve in domain s.

$$(sX - x_0) = Y \tag{94}$$
$$(sY - y_0) = -X + 2Y \tag{95}$$

From (94) we can express X and substitute into (95).

$$\begin{aligned}
X &= (Y + x_0)/s \\
(sY + y_0) &= -\left(\frac{Y + x_0}{s}\right) + 2Y \\
sY - y_0 &= -\frac{Y}{s} - \frac{x_0}{s} + 2Y \\
sY + \frac{Y}{s} - 2Y &= y_0 - \frac{x_0}{s} \\
Y(s + \frac{1}{s} - 2) &= \frac{y_0 s - x_0}{s} \\
Y(s^2 + 1 - 2s) &= y_0 s - x_0 \\
Y &= \frac{y_0 s - x_0}{(s-1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} \qquad (96) \\
y_0 s - x_0 &= A(s-1) + B \\
y_0 s - x_0 &= As - A + B \\
y_0 s - x_0 &= s(A) + 1(-A + B) \implies \\
\implies A &= y_0,\ B = -x_0 + y_0 \\
Y &= \frac{y_0}{s-1} + \frac{-x_0 + y_0}{(s-1)^2}
\end{aligned}$$

The method of undetermined coefficients was used to find $A,\ B$. The solution $y(t)$ is inverse transform of $Y(s)$:

$$\underline{\underline{y(t) = e^t y_0 + te^t(y_0 - x_0)}}.$$

There are more ways how to find $x(t)$. This time (96) is substituted into (94):

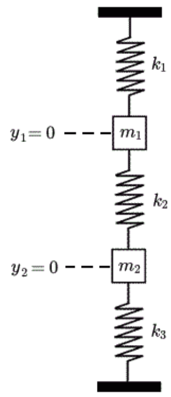
$$
\begin{aligned}
sX - x_0 &= \frac{y_0 s - x_0}{(s-1)^2} \\
sX &= \frac{y_0 s - x_0}{(s-1)^2} + x_0 \frac{s^2 - 2s + 1}{(s-1)^2} \\
sX &= \frac{y_0 s - x_0 + x_0 s^2 - x_0 \cdot 2s + x_0}{(s-1)^2} \\
sX &= \frac{y_0 s + x_0 s^2 - x_0 \cdot 2s}{(s-1)^2} \\
X &= \frac{y_0 + x_0 s - 2x_0}{(s-1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} \\
y_0 + x_0 s - 2x_0 &= A(s-1) + B \\
y_0 + x_0 s - 2x_0 &= As - A + B \\
y_0 + x_0 s - 2x_0 &= s(A) + 1(-A + B) \implies \\
&\implies A = x_0,\ B = y_0 - x_0
\end{aligned}
$$

Again the complicated fraction was decomposited into two simpler fractions, method of undetermined coefficients was used to determine $A,\ B$.

$$\underline{\underline{x(t) = e^t x_0 + te^t (y_0 - x_0)}}$$

Example

Derive the system of differential equations describing the motion of two masses according to the picture. Use Laplace transform to solve if $k_1 = k_2 = k_3 = 1,\ m_1 = m_2 = 1$ and initial values $y_1(0) = 0,\ y_1'(0) = -1,\ y_2(0) = 0,\ y_2'(0) = 1$.



The initial conditions say that both bodies are at their equilibrium point and have initial speed.

We can borrow general description of spring motion from chapter modeling: $M\ddot{u} + C\dot{u} + Ku = f$. In this example is no dumping involved, no external force. We are used to use variable y instead of u.

$$
\begin{aligned}
my'' + ky &= 0 \qquad \text{(general form)} \\
m_1 y_1'' + k_1 y_1 + k_2(y_1 - y_2) &= 0 \\
m_2 y_2'' + k_2(y_2 - y_1) + k_3 y_2 &= 0
\end{aligned}
$$

When the masses and spring constant are substituted:

$$
\begin{aligned}
y_1'' + y_1 + y_1 - y_2 &= 0 \\
y_2'' + y_2 - y_1 + y_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y_1'' + 2y_1 - y_2 &= 0 \\
y_2'' + 2y_2 - y_1 &= 0
\end{aligned}
$$

We have a system of DE which will be solved by Laplace. Let us transform each member and solve in the domain s.

$$s^2Y_1 - s\cdot 0 - (-1) + 2Y_1 - Y_2 = 0$$
$$s^2Y_2 - s\cdot 0 - 1 + 2Y_2 - Y_1 = 0$$

$$s^2Y_1 + 1 + 2Y_1 - Y_2 = 0$$
$$s^2Y_2 - 1 + 2Y_2 - Y_1 = 0$$

$$Y_1(s^2+2) + 1 - Y_2 = 0$$
$$Y_2(s^2+2) - 1 - Y_1 = 0 \quad (97)$$

$$Y_1 = \frac{Y_2 - 1}{s^2+2} \quad (98)$$
$$Y_2 = \frac{Y_1 + 1}{s^2+2} \quad (99)$$

Now (98) is substituted into (97).

$$Y_2(s^2+2) - 1 - \frac{Y_2 - 1}{s^2+2} = 0$$
$$Y_2s^2 + 2Y_2 - 1 - \frac{Y_2 - 1}{s^2+2} = 0$$
$$Y_2s^4 + 2Y_2s^2 - s^2 + 2Y_2s^2 + 4Y_2 - 2 - Y_2 + 1 = 0$$
$$Y_2s^4 + 4Y_2s^2 + 3Y_2 - s^2 - 1 = 0$$
$$Y_2(s^4 + 4s^2 + 3) = 1 + s^2$$
$$Y_2 = \frac{1+s^2}{(s^2+1)(s^2+3)}$$
$$Y_2 = \frac{1}{s^2+3}$$

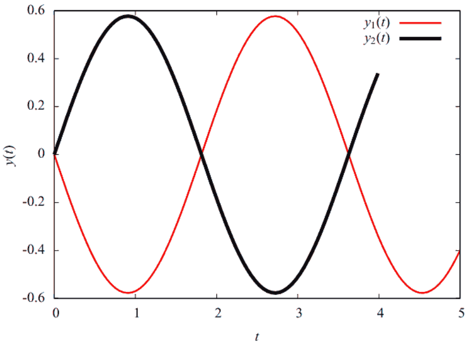
The last line can be substituted into (98):

$$Y_1 = \frac{\frac{1}{s^2+3} - 1}{s^2+2} = \frac{\frac{1-s^2-3}{s^2+3}}{s^2+2} = \frac{\frac{-s^2-2}{s^2+3}}{s^2+2} = \frac{-1}{s^2+3}$$

The solution of the system of DE is done by inverse Laplace transform of $Y_1(s)$ and $Y_2(s)$.

$$\underline{\underline{y_1(t) = -\sin\sqrt{3}t \cdot \frac{1}{\sqrt{3}}}}$$

$$\underline{\underline{y_2(t) = \sin\sqrt{3}t \cdot \frac{1}{\sqrt{3}}}}$$

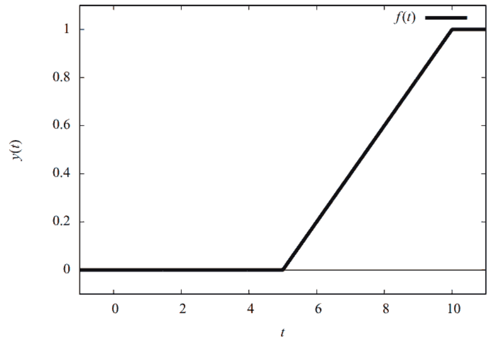


The response of the system

Example

Solve

$$y'' + 4y = f(t), \qquad y(0) = 0,\ y'(0) = 0 \qquad f(t) = \begin{cases} 0, & t < 5, \\ (t-5)/5, & 5 \le t < 10, \\ 1, & t \ge 10. \end{cases}$$



The physical meaning of $f(t)$ is known as ramp loading

The first step is to describe the given function $f(t)$ by means of unit step function, otherwise we can not make Laplace transform of $f(t)$.

$$\begin{aligned} f(t) &= u_5\left(\frac{t-5}{5}\right) - u_{10}\left(\frac{t-5}{5}\right) + u_{10}(1) = \\ &= \frac{1}{5}\,u_5(t-5) + u_{10}\left(-\frac{t-5}{5} + \frac{5}{5}\right) = \\ &= \frac{1}{5}\,u_5(t-5) - \frac{1}{5}\,u_{10}(t-10) \end{aligned}$$

The next step is usual: we have to transform each term of DE. The members of function $f(t)$, which involves unit step time functions will be transformed the using shifting theorem (92).

$$\begin{aligned} (s^2Y - s\cdot 0 - 0) + 4Y &= \frac{1}{5}\,\frac{1}{s^2}\,e^{-5s} - \frac{1}{5}\,\frac{1}{s^2}\,e^{-10s} \\ Y(s^2+4) &= \frac{\frac{1}{5}\,e^{-5s} - \frac{1}{5}\,e^{-10s}}{s^2} \\ Y &= \frac{\frac{1}{5}\,e^{-5s} - \frac{1}{5}\,e^{-10s}}{s^2(s^2+4)} \end{aligned}$$

Both the terms in numerator express time shift, not the functions themselves. Thus we can bring time shift outside of the fraction and bring them back later.

$$Y = \left(\frac{1}{5}\,e^{-5s} - \frac{1}{5}\,e^{-10s}\right)\frac{1}{s^2(s^2+4)}$$

We have to do partial fraction decomposition of $\frac{1}{s^2(s^2+4)}$:

$$\begin{aligned} \frac{1}{s^2(s^2+4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} \\ 1 &= A(s)(s^2+4) + B(s^2+4) + Cs(s^2) + D(s^2) \\ 1 &= s^3(A+C) + s^2(B+D) + s(4A) + 1(4B) \implies \\ B &= \frac{1}{4},\ A = 0,\ C = 0,\ D = -\frac{1}{4} \end{aligned}$$

Then we have to do inverse transform of $Y(s)$:

$$Y = \left(\frac{1}{5}\,e^{-5s} - \frac{1}{5}\,e^{-10s}\right)\left(\frac{1}{4}\,\frac{1}{s^2} - \frac{1}{8}\,\frac{2}{s^2+4}\right)$$

The inverse terms inside the second pair of parentheses are $\frac{1}{4}\,t - \frac{1}{8}\sin 2t$ and the inverse transform of $Y(s)$ is

$$y(t) = \left(\frac{1}{5}\,u_5(t) - \frac{1}{5}\,u_{10}(t)\right)\left(\frac{1}{4}\,t - \frac{1}{8}\sin 2t\right) =$$
$$= \frac{1}{5}\,u_5(t)\left(\frac{1}{4}\,t - \frac{1}{8}\sin 2t\right) - \frac{1}{5}\,u_{10}(t)\left(\frac{1}{4}\,t - \frac{1}{8}\sin 2t\right)$$

- Note that both $\frac{1}{4}\,t$ and $\frac{1}{8}\sin 2t$ are erased by unit step function: until 5 s there is a zero response.
- At time $t = 5$ both members $\frac{1}{4}\,t$ and $\frac{1}{8}\sin 2t$ become active (we also have to shift the functions according to the shifting theorem): $y(5 \le t < 10) = \frac{1}{5}\cdot\frac{1}{4}\,(t-5) - \frac{1}{5}\cdot\frac{1}{8}\sin 2(t-5)$.
- At time $t = 10$ we have to subtract $\frac{1}{5}\left(\frac{1}{4}\,(t-10) - \frac{1}{8}\sin 2(t-10)\right)$

$$y(t) = \begin{cases} 0, & t < 5, \\ \frac{1}{20}\,(t-5) - \frac{1}{40}\sin 2(t-5), & 5 \le t < 10 \\ \frac{5}{20} - \frac{1}{40}\sin 2(t-5) + \frac{1}{40}\sin 2(t-10), & t \ge 10. \end{cases}$$

That is the final answer.

Example

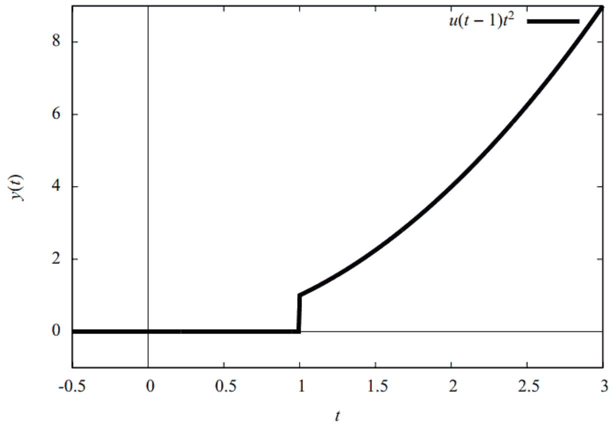
Find the solution $y(t)$ if

$$Y(s) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

An event is happening at time $t = 1$. According to the shifting theorem, it can be observed that each function is time-shifted into $t = 1$. Since they are shifted, they are also erased.

$$y(t) = u_1(t)\left((t-1)^2 + 2(t-1) + 1\right) = u_1(t)(t^2 - 2t + 1 + 2t - 2 + 1) = u_1(t)\cdot t^2.$$

What seemed as a time shift is finally t^2 erased until $t = 1$. Indeed, such example has been computed above already.



Example

Find the solution $y(t)$ if

$$Y(s) = \frac{2}{s-1} - \frac{1}{s} - \frac{2}{s}\,e^{-3s} - \frac{1}{s^2}\,e^{-3s} + \frac{3/2}{s-1}\,e^{-3s} + \frac{1/2}{s+1}\,e^{-3s}$$

We can recognize some terms which are active from $t = 0$ and some other terms shifted and active from time $t = 3$:

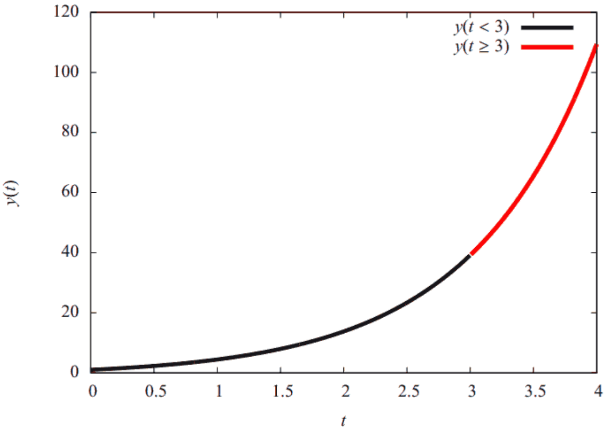
$$y(t) = 2e^t - 1 - 2u_3(t) - u_3(t) \cdot (t-3) + \frac{3}{2}e^{t-3}u_3(t) + \frac{1}{2}e^{-(t-3)}u_3(t)$$

Such solution is correct but not convenient and it is expected we provide also the case form:

$$y(t) = \begin{cases} -1 + 2e^t & 0 \le t < 3, \\ -1 + 2e^t - 2 - (t-3) + \frac{3}{2}e^{t-3} + \frac{1}{2}e^{-(t-3)} & t \ge 3, \end{cases}$$

which can be simplified to

$$y(t) = \begin{cases} -1 + 2e^t & 0 \le t < 3, \\ 2e^t - t + \frac{3}{2}e^{t-3} + \frac{1}{2}e^{3-t} & t \ge 3. \end{cases}$$



Example

Find the solution $y(t)$ if

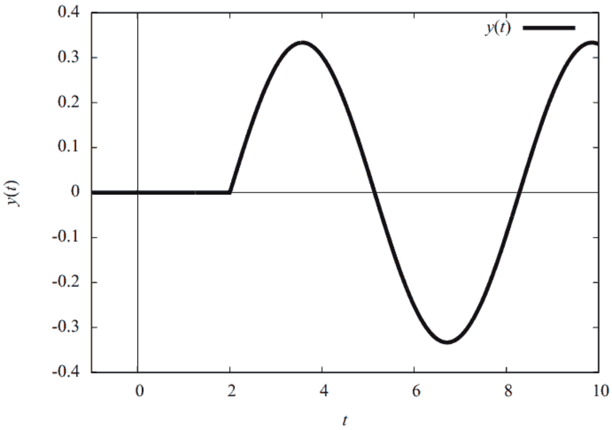
$$Y(s) = \frac{e^{-2s}}{s^2 + 9}$$

It can be observed that sine is involved, more precisely $\sin 3t$ which has been shifted to $t = 2$.

$$y(t) = \frac{1}{3}\sin 3(t-2) \cdot u_2(t)$$

So the solution is $1/3 \sin 3t$ moved to $t = 2$. The case form is more convenient:

$$y(t) = \begin{cases} 0 & t \le 2, \\ \frac{1}{3}\sin 3(t-2) & t > 2. \end{cases}$$



Example

Find the solution $y(t)$ if

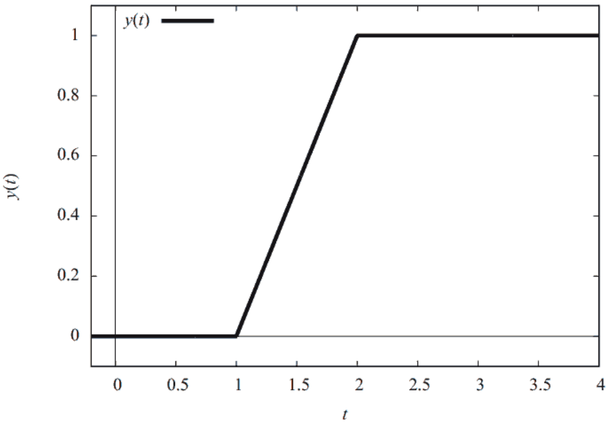
$$Y(s) = \frac{1}{s^2} e^{-s} - \frac{1}{s^2} e^{-2s}$$

We can identify two functions $f(t) = t$ which were moved to time $t = 1$ and $t = 2$

$$f(t) = (t-1)u(t-1) - (t-2)u(t-2)$$

Such result, written with unit step function, is formally correct, but for convenience its case form is expected:

$$f(t) = \begin{cases} 0 & t < 1, \\ t-1 & 1 \le t < 2, \\ 1 & t \ge 2. \end{cases}$$



Laplace transform

Additional properties and operations

As was told in the introduction, Laplace transform can handle e.g. impulse or periodic function as the driving function. We are going to cover some more tools and operations here.

Laplace transform of functions multiplied by t^n

We have experience that solving DE may led to solution which has to be prefixed by $t,\ t^2,\ \ldots$. For example $y^{(4)} - 3y'' + 2y' = 0$ has solution $y_c = c_1 + c_2 e^t + c_3 t e^t - c_4 e^{-2t}$.

That might happen also to solutions based on $\sin(), \cos()$ as well. Such functions might appear not only within the solution, but also on the right side of DE.

It was told that $\mathscr{L}[e^{at} t^n] = n!/(s-a)^{n+1}$ and the general rule is

$$\text{If } F(s) = \mathscr{L}[f(t)], \text{ then}$$

$$\mathscr{L}[t^n \cdot f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) \tag{100}$$

Example

Find $\mathscr{L}[t \cdot e^t]$ and $\mathscr{L}[t^2 \cdot e^t]$ using the rule (100).

If $f(t) = e^t$ then the Laplace transform is $F(s) = 1/(s-1)$. That is easily solved or the tables can be used. Now let us find $\mathscr{L}[t \cdot e^t]$ and $\mathscr{L}[t^2 \cdot e^t]$. For that purpose, let us differentiate $F(s)$ once and twice.

$$\underline{\underline{\mathscr{L}\left[t \cdot e^t\right]}} = (-1)\frac{d}{ds} \cdot \frac{1}{s-1} = \underline{\underline{\frac{1}{(s-1)^2}}}$$

$$\underline{\underline{\mathscr{L}\left[t^2 \cdot e^t\right]}} = (-1)^2 \frac{d^2}{ds^2} \cdot \frac{1}{(s-1)^2} = 2\,\underline{\underline{\frac{1}{(s-1)^3}}}$$

Convolution

Solving DE by Laplace transform sometimes leads to such **fractions which are not easily solved by partial fraction decomposition. We would like to take the inverse way of such transforms.**

For example $\frac{1}{s(s^2+1)}$ might be not obvious how to be solved and then convolution theorem is a handy tool.

The convolution is defined by the following integral.

$$f * g = g * f = \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau. \tag{101}$$

The convolution theorem states that

$$\mathscr{L}\left[f(t)\right] \cdot \mathscr{L}\left[g(t)\right] = F(s) \cdot G(s) = \mathscr{L}\left[f * g\right],$$

and we are more interested into the inverse form of the theorem:

$$\mathscr{L}^{-1}\left[F(s) \cdot G(s)\right] = f * g \tag{102}$$

Example

Use the convolution theorem to find inverse transform of $F(s) = \frac{1}{s(s^2+1)}$.

We can detect two fractions, which are products of $f(t) = \sin(t)$ and $g(t) = 1$. But this time they are multiplied together and how to separate them into two fractions? If we do not know, we may feed them according to (102) into (101).

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s}\right] = \left|\begin{matrix} f(t)=\sin t \\ g(t)=1 \end{matrix}\right| = f*g$$

$$\int_{\tau=0}^{t} \sin\tau\cdot 1\, d\tau = -\cos\tau|_0^t = -\cos t + \cos 0 \implies$$

$$\implies \mathscr{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1-\cos t$$

Note that the result could be also achieved by partial fraction decomposition if one knows all the rules.

$$F(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$
$$1 = A(s^2+1) + Bs^2 + Cs$$
$$1 = s^2(A+B) + s(C) + 1(A) \implies A=1,\ B=-1,\ C=0 \implies$$
$$\implies \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

And so we have approached the same result of inverse transform

$$\mathscr{L}^{-1}\left[F(s)\right] = 1-\cos t.$$

Transform of periodic function

In many applications the nonhomogeneous term in a linear DE is a periodic function. For example $\sin(t)$ and $\cos(t)$ are periodic function with period of $T = 2\pi$. The Laplace transform of periodic function is

$$\mathscr{L}[f(t)] = \frac{\mathscr{L}[f_T(t)]}{1-e^{-sT}} \qquad \text{where} \tag{103}$$
$$f_T(t) = \begin{cases} f(t) & 0 \le t \le T, \\ 0 & t > T \end{cases}$$

So it is Laplace transform of the first period divided by $1 - e^{-sT}$

Example

Determine the Laplace transform of function

$$f(t) = \begin{cases} 1 & 0 \le t \le T/2 \\ 0 & T/2 < t < T \end{cases} \qquad f(t+T) = f(t),\ t \ge 0$$

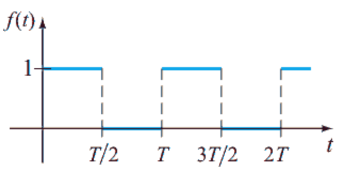


FIGURE 7.4.4 Square wave

The function is periodic with a period T. We have to evaluate Laplace transform of the first period and then use theorem (103).

$$\mathscr{L}[f_T(t)] = \int_0^T f(t)e^{-st}\,dt = \int_0^{T/2} 1\cdot e^{-st}\,dt =$$
$$= -\frac{1}{s}\,e^{-st}\Big|_{t=0}^{T/2} = -\frac{1}{s}\,e^{-sT/2} - (-\frac{1}{s}\,e^{0}) = \underline{\frac{1-e^{-sT/2}}{s}}$$

The Laplace transform of the first period was evaluated by means of integral. Now substitute to (103):

$$\mathscr{L}[f(t)] = \frac{\frac{1-e^{-sT/2}}{s}}{1-e^{-sT}} = \frac{1-e^{-sT/2}}{s(1-e^{-sT})}$$
$$\underline{\underline{\mathscr{L}[f(t)] = \frac{1}{s(1+e^{-sT/2})},}} \qquad s>0$$

The last simplification comes from $\frac{1-a}{1-a^2} = \frac{1}{1+a}$.

The Dirac delta function

The other useful property of Laplace transform is that it can apply an impulse as a driving function $f(t)$. It can be either force, voltage or some other physical phenomena.

We want to have a function which in general has a value of zero for any t except the peak within requested short timeframe. As we have e.g. unit time function already, we would like if such "peak" function is *unit* also, namely the area below the function is 1. Such function δ was introduced by Paul Dirac in 1930. The function is defined as

$$\delta(t-t_0) = \begin{cases} \infty & t = t_0, \\ 0 & t \neq t_0, \end{cases} \quad \text{and} \qquad (104)$$
$$\int_0^\infty \delta(t-t_0)\,dt = 1$$

In fact, nothing actually behaves like this, but it serves well for solving problems. From purely mathematical viewpoint, the Dirac delta is even not strictly a function.

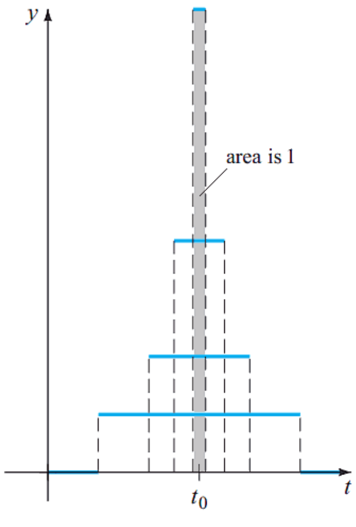


FIGURE 7.5.2 Unit impulse

The impulse proceeds in a very short timeframe, let us say in 0.1 s and the value of $\delta(t-t_0)$ is then 10 (in time $t=t_0$). If we consider the timeframe shorter, the value of impulse has to grow, since it is a unit impulse (area = 1). If the effect occurs "instantaneously" at $t=t_0$ it has an infinite magnitude.

Laplace transform of Dirac delta can be found from definition. Let us evaluate Laplace transform of $f(t)\,\delta(t-t_0)$ and transform of Dirac delta itself is a case if $f(t)=1$ then:

$$\mathscr{L}[f(t)\,\delta(t-t_0)] = \int_0^\infty e^{-st}\,f(t)\,\delta(t-t_0)\,dt = e^{-st}\,f(t)\int_0^\infty \delta(t-t_0)\,dt \implies$$
$$\implies \mathscr{L}[f(t)\,\delta(t-t_0)] = e^{-st}\,f(t) \qquad (105)$$

The integral is laid from zero to infinity. However δ has a value of zero anywhere except its peak. That implies that the limits of integral can be reduced to the vicinity of t_0. And the interval is so small that the value of functions $e^{-st} f(t)$ are kept constant, therefore can be moved in front of integral. And what is left inside integral is known to be 1 (from (104)). So the Laplace transform of $f(t)\,\delta(t-t_0)$ is $e^{-st} f(t)$.

Example

A mass of 1 kg is attached to a spring with spring constant $k=1$ N/m. The mass is at rest. At $t=0$ a hammer blows an impulse 1 to the mass and the second blow at $T=2\pi$. What is the response of the system?

We have some idea how the system behaves from the chapter modeling. We know that the DE of mass on the spring is described as $M\ddot{u}+C\dot{u}+Ku=f$. There is no damping in this example considered, so let us use the equation in form

$$my''+ky=f(t), \qquad y(0)=0,\ y'(0)=0$$

After substituting $m=1$, $k=1$ and $f(t)=\delta(t-0)+\delta(t-2\pi)$ we have DE to solve.

$$\begin{aligned} y''+y&=\delta(t)+\delta(t-2\pi) \\ s^2Y+Y&=e^{-0}+e^{-2\pi s} \\ Y(s^2+1)&=1+e^{-2\pi s} \\ Y&=\frac{1+e^{-2\pi s}}{s^2+1}=\frac{1}{s^2+1}+\frac{e^{-2\pi s}}{s^2+1} \end{aligned}$$

The first fraction can be identified as a product of $\sin t$. The second is also $\sin t$ and term $e^{-2\pi s}$ records time shift at $t=2\pi$.

$$y(t)=\sin t+u_{2\pi}(t)\sin(t-2\pi)=\sin t+u_{2\pi}(t)\sin(t)$$

The final answer has to be written in the case form:

$$y(t)=\begin{cases} \sin t & 0\le t\le 2\pi \\ 2\sin t & t>2\pi \end{cases}$$

Example

Nonhomogeneous member on the right side of DE is $f(t)=2\delta(t-\pi)-\delta(t-2\pi)$. Find the transformed function $F(s)$.

We are going to transform impulse (Dirac delta function). There are two impulses, the first one at $t=\pi$, the second one at $t=2\pi$. These impulses last for infinitesimally small timeframe and they are of unit size (area is 1 when integrated). That means if we use definition of Laplace transform ($\int_0^\infty e^{-st}\delta(t_0)\,dt$), the member e^{-st} anywhere where delta function is *nonzero* can be considered as not changing and can be taken as e^{-st_0} in front of the integral. Then e^{-st_0} has to be multiplied by integral. Only delta function remains in integral and it is known from the definition of Dirac function that the value of integral is one.

$$F(s)=e^{-\pi s}+e^{-2\pi s}$$

Example

Nonhomogeneous member on the right side of DE is $f(t)=\delta(t-2\pi)\cos t$. Find the transformed function $F(s)$.

The task is to transform $\delta(t-2\pi)\cos t$. If we use the definition of Laplace transform, we have $\int_0^\infty e^{-st}\sin t\,\delta(t_0)\,dt$. Anywhere except at $t_0=2\pi$ the Dirac delta function has value of zero. Whatever makes any sense to multiply happens at $t=2\pi$. Since it is at single point, $e^{-st}\sin t$ is not changing and can be taken outside of the integral. And the value of integral is known to be *one* from the definition of the Dirac function.

$$F(s)=e^{-2\pi s}\cos 2\pi=e^{-2\pi s}$$

Example

Find the first and the second antiderivative of $\delta(t-1)$.

For the first antiderivative we have to solve

$$y' = \delta(t-1), \qquad y(0) = 0$$

Let us transform each member and solve:

$$\begin{aligned} sY - y(0) &= e^{-s} \\ sY &= e^{-s} \\ Y(s) &= \frac{e^{-s}}{s} \end{aligned}$$

The solution is then

$$\underline{\underline{y(t) = 1 \cdot u_1(t) = u_1(t)}}$$

For the second antiderivative we have to solve

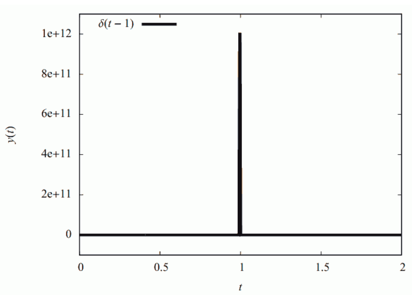
$$y'' = \delta(t-1), \qquad y(0) = 0,\ y'(0) = 0$$

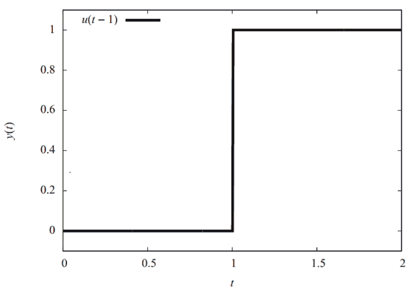
Let us transform each member and solve:

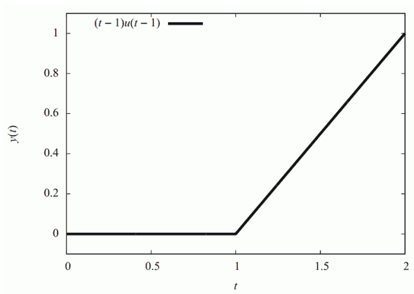
$$\begin{aligned} s^2Y - sy(0) - y'(0) &= e^{-s} \\ s^2Y &= e^{-s} \\ Y(s) &= \frac{e^{-s}}{s^2} \end{aligned}$$

The solution is then

$$\underline{\underline{y(t) = (t-1) \cdot u_1(t)}}$$







Laplace transform

Solving partial DE (PDE)

The task arises from physical situations (heat equation; distribution of heat in a given region over time). We have function u of two variables: $u(x,t)$, where the variable t represents time $t \geq 0$, the other variable x is a location. Laplace transform of $u(x,t)$ is $\mathscr{L}[u(x,t)] = \int_0^\infty u(x,t)e^{-st}\,dt$, where paramater x is treated as a constant. We still use capital letter to denote Laplace transform of a given function:

$$\mathscr{L}[u(x,t)] = U(x,s) = U$$

Since differential equation to solve can look like (examples)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x \qquad \text{or}$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = f(x),$$

we need Laplace transforms of corresponding derivatives—and these are analogy to Laplace transform of function $y(t)$.

$$\mathscr{L}\left[u_t(x,t)\right] = \mathscr{L}\left[\frac{\partial u}{\partial t}\right] = sU(x,s) - u(x,0), \tag{106}$$
$$\mathscr{L}\left[u_{tt}(x,t)\right] = \mathscr{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2U(x,s) - su(x,0) - u_t(x,0). \tag{107}$$

Since we are transforming with respect to t, we further suppose it is legitimate to move $\partial u/\partial x$ to front of integral:

$$\mathscr{L}\left[u_x(x,t)\right] = \mathscr{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^\infty e^{-st}\frac{\partial^2 u}{\partial x^2}\,dt = \frac{d^2}{dx^2}\int_0^\infty e^{-st}u(x,t)\,dt = \frac{d^2}{dx^2}\mathscr{L}\left[u(x,t)\right]$$

So we will put

$$\mathscr{L}\left[\frac{\partial u}{\partial x}\right] = \frac{dU}{dx}, \tag{108}$$
$$\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2 U}{dx^2}. \tag{109}$$

The PDE will be converted into ordinary DE (ODE).

Example

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x \qquad x > 0,\ t > 0, \qquad u(x=0,t) = 0,\ u(x,t=0) = 0$$

We are given a partial differential equation (PDE). We solve by Laplace, so we have to transform each term. **Transform is made with respect to time t, the other dimension x is considered to be a constant.** To transform $\partial u/\partial t$, it is an analogy with $y'(t) = dy/dt$ and we use (106). For $\partial u/\partial x$, we drag $\partial/\partial x$ out of the integral, because the transform proceeds over t, while differentiation is with respect to x and then we have—according to (108)—$\mathscr{L}[\partial u/\partial x] = d/dx\,U$.

$$\frac{dU}{dx} + sU(x,s) - u(x,0) = x \cdot \frac{1}{s}$$
$$\frac{dU}{dx} + sU(x,s) = x \cdot \frac{1}{s}$$

$u(x,0) = 0$ is given by boundary/initial conditions. We have converted PDE into ODE: the last equation can be solved as linear DE. Now dependent variable is U, independent is x. We are solving $U(x)$, s is considered to be a constant.

$$U' + sU = x\,\frac{1}{s} \implies P(x) = s \implies \mu(x) = e^{\int P(x)\,dx} = e^{\int s\,dx} = e^{sx}$$

The linear DE are solved by identifying $P(x)$ in order to express integrating factor $\mu(x) = e^{\int P(x)\,dx}$. The integrating factor $\mu(x)$ is used to multiply DE and then it is easy to integrate, because on the left side is $d/dx\ \mu(x)y(x)$ or in current case $d/dx\ \mu(x)U(x)$.

$$\begin{aligned} e^{sx}U' + e^{sx}sU &= s^{sx}x\,\frac{1}{s} \\ \frac{d}{dx}\,e^{sx}U &= e^{sx}x\,\frac{1}{s} \\ e^{sx}U &= \frac{1}{s}\cdot\frac{(sx-1)e^{sx}}{s^2} + c \\ U(x,s) &= \frac{1}{s}\cdot\frac{sx-1}{s^2} + ce^{-sx} = \frac{x}{s^2} - \frac{1}{s^3} + ce^{-sx} \end{aligned}$$

The integral of $e^{sx}x$ is solved by integrating by parts. We use the product rule for differentiation backwards.

$$(uv)' = u'v + uv' \implies u'v = (uv)' - uv' \implies \int u'v = uv - \int uv'$$

Then we have to locate one term, which is easy to differentiate (x) and the second one, which is easy to integrate (e^{sx}).

$$\int e^{sx}x\,dx = \left| \begin{matrix} u' = e^{sx} & u = \frac{1}{s}\,e^{sx} \\ v = x & v' = 1 \end{matrix} \right| = \frac{1}{s}\,e^{sx}x - \int \frac{1}{s}\,e^{sx}\,dx = x\,\frac{1}{s}\,e^{sx} - \frac{1}{s^2}\,e^{sx} =$$
$$= \frac{(xs-1)e^{sx}}{s^2}$$

We are used to evaluate arbitrary constants as c here (product of integration) by applying initial/boundary values:

$$u(x=0,t) = 0 \implies U(x=0,s) = 0 \implies 0 = -\frac{1}{s^3} + c \implies c = \frac{1}{s^3} \implies$$
$$\implies U(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{1}{s^3}\,e^{-sx}$$

We reached the solution but it is in s domain. Finally we have to go through inverse Laplace transform.

$$\underline{\underline{u(x,t) = xt - \frac{1}{2}\,t^2 + u(t-x)\cdot\frac{1}{2}\,(t-x)^2}}$$

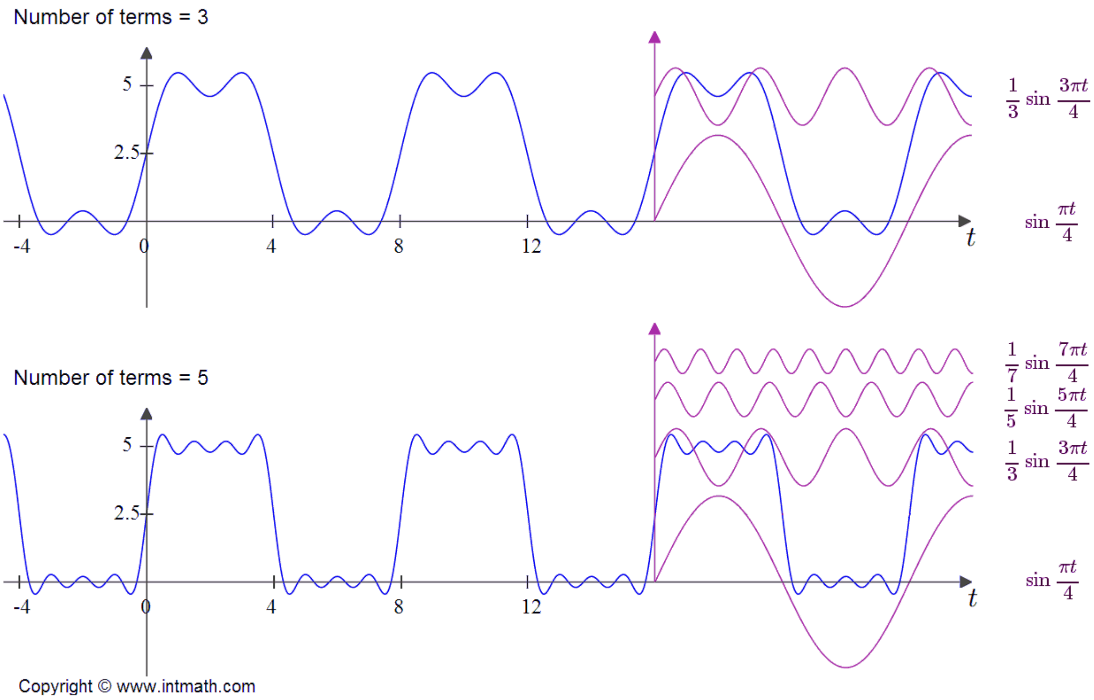
Note: here notation $u(t-x)$ is equivalent of $u_x(t)$ and is unit step function (the product of inverse Laplace transform of e^{-sx}). The above solution is formally correct, but we should rather use the case form. The first two terms do not depend on any unit step function and are valid from zero to infinity. The last term, because of unit step function $u_x(t)$, is inactive until $t = x$:

$$u(x,t) = \begin{cases} xt - \frac{1}{2}\,t^2 & t < x \\ xt - \frac{1}{2}\,t^2 + \frac{1}{2}\,(t-x)^2 & t \geq x \end{cases}$$

That can be further simplified to

$$u(x,t) = \begin{cases} xt - \frac{1}{2}\,t^2 & t < x \\ \frac{1}{2}\,x^2 & t \geq x \end{cases}$$

Fourier serier



Approximation of the square function by Fourier series

Fourier series is a way to represent a function as a combination of simple sine waves. More formally, it decomposes any periodic function or periodic signal into a sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines. Fourier series does not try to approximate given function around interested point but approximates the whole interval.

With Fourier series, any function, which is periodic on 2π, can be expressed as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt. \tag{110}$$

Fourier series for solving DE

Suppose, that the input function of DE is $\sin nt$. Then we are able to calculate response $y(t)$. Fourier series is useful tool to solve **linear DE** (e.g. $y'' + a_1 y' + a_0 y = f(t)$). If the solution can be expressed for one sine, then, because of **linearity and superposition,** the method is applicable also for Fourier series. The advantage is there are **no restrictions for the function $f(t)$ on the right side.** Recall the other methods which have limitation of the nonhomogeneous member: they have to be expressed in terms as e^t, $\sin t$ and so on.

We want to calculate Fourier series for a given function $x(t)$, which is periodic on 2π.

Orthogonality of functions

In order to derive formula expressing coefficients a_n, b_n in (110) we have to be familiar with orthogonality of functions.

Two functions are orthogonal on an interval $[a, b]$ if their inner product is zero:

$$\int_a^b f_1(t) f_2(t)\, dt = 0$$

The set of functions $\{f_0(t), f_1(t), \dots\}$ is orthogonal on an interval $[a, b]$ if

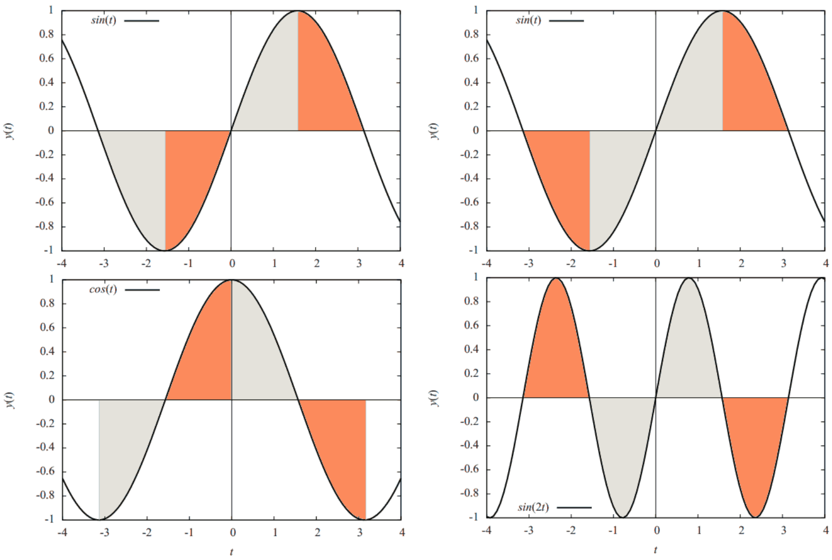
$$\int_a^b f_m(t) f_n(t)\, dt = 0 \quad \text{for each } m \neq n$$

Orthogonal has meaning similar to perpendicular, however we do not use such term. In this chapter we are more interested into orthogonality of sines and cosines on interval $[-\pi, \pi]$. It can be shown, that if we have a set of functions S

$$S = \begin{cases} \cos nt & n = 0, 1, \dots, \infty \\ \sin mt & m = 1, 2, \dots, \infty, \end{cases}$$

then any distinct combination from the set is orthogonal (their inner product is zero) on $[-\pi, \pi]$. Only

$$\left.\begin{aligned} \int_{-\pi}^{\pi} \sin^2 nt \, dt \\ \int_{-\pi}^{\pi} \cos^2 nt \, dt \end{aligned}\right\} = \pi$$



To illustrate the orthogonality of the functions mentioned. It should be obvious, that the inner product in both above examples is zero ($\int_{-\pi}^{\pi} \sin t \; \cos t \, dt = 0$, $\int_{-\pi}^{\pi} \sin t \; \sin 2t \, dt = 0$).

So now we have a task to find values of coefficients a_n, b_n for a given function $f(t)$ of 2π period. Let us find a_n for cosines, because b_n is the same logic.

$$f(t) = \dots + a_k \cos kt + \dots + a_n \cos nt + \dots$$

In the above expression $a_n \cos nt$ is the term which we are studying—we want to determine a_n. The other term is just *some other* term. Now let us multiply each member by $\cos nt$.

$$\cos nt \; f(t) = \dots + a_k \cos kt \cdot \cos nt + \dots + a_n \cos^2 nt + \dots$$

And finally let us integrate on interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \cos nt \; f(t) \, dt = \dots + \int_{-\pi}^{\pi} a_k \cos kt \cdot \cos nt \, dt + \dots + \int_{-\pi}^{\pi} a_n \cos^2 nt \, dt + \dots$$

Because of orthogonality all *some other* members are zero, so we can express a_n

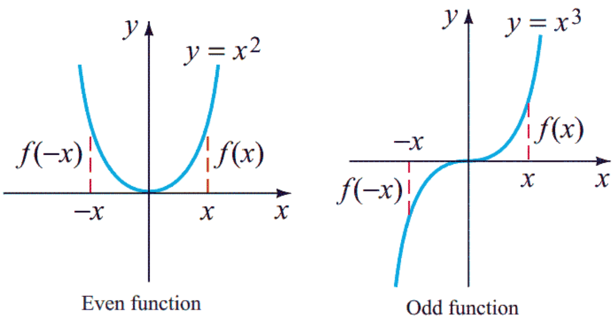
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \qquad (111)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \qquad (112)$$

Tools helpful to shorten the calculations

Function is odd or even

If the function is odd or even, there are only sines or cosines within series.

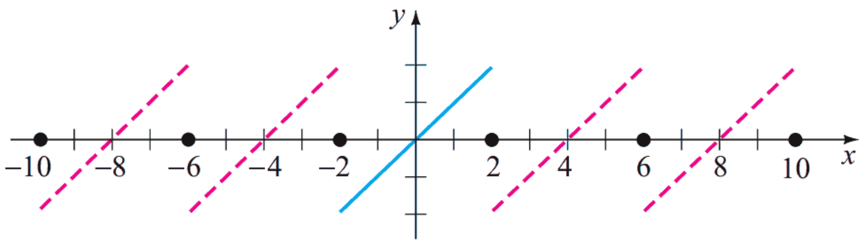


*Function is **even:** Fourier series consists only of cosine terms, all b_n are zero.*
*Function is **odd:** Fourier series consists only of sine terms, all a_n are zero.*

If function is odd or even, it is **better to integrate on half-interval** $[0, \pi]$ instead of $[\pi, \pi]$ **and double the result.** In the case of cosine it is obvious why we can use the double result of integral $\int_0^\pi$. In the case of sine, both sine and function $f(t)$ are either positive or negative, so the result has to be positive also.

Convergence of Fourier series

The series converges to $f(t)$ except the points where the function is not continuous. If there is a discontinuity at t_0, the series converges to the midpoint.



Fourier series converges to $f(t)$ except the points of discontinuity. In such points it converts to the middle.

Extensions

Period other than 2π

It would be not much useful if we could work only with functions with period of 2π. Instead of half-period π we will have p (the period is $2p$). Instead of $\cos nt$ we have $\cos nt\,\frac{\pi}{p}$ and for the series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt\,\frac{\pi}{p} + b_n \sin nt\,\frac{\pi}{p}\,. \tag{113}$$

the rules to express coefficients are

$$a_n = \frac{1}{p}\int_{-p}^{p} x(t)\cos nt\,\frac{\pi}{p}\;dt \tag{114}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} x(t)\sin nt\,\frac{\pi}{p}\;dt \tag{115}$$

Given function is not periodic

Even if the function is defined on interval $(0, p)$, but is not periodic, we can use Fourier series. The function can be extended either to an odd or even function.

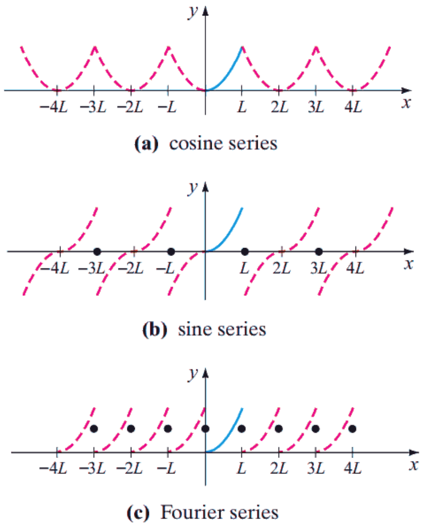


FIGURE 11.3.11 Same function on $(0, L)$ but different periodic extensions

Example

Find the Fourier series of square 2π periodic wave defined on interval $[-\pi, \pi]$ as

$$x(t) = \begin{cases} 0 & -\pi < t \leq 0, \\ 1 & 0 < t \leq \pi. \end{cases}$$

See also red line on the below graph.

We have a case with a period of 2π. In order to collect serier (110) we have to evaluate a_0, a_n, b_n. It can be observed from the graph of given periodic function that $a_0 = 1$, because the first term of series (mean, average) has to be $a_0/2 = 0.5$. Then it is clear that the given function is odd.

If the function is odd, all a_n associated with cosines remain zero, the cosine members are inactive, cosines are not helpful in describing the function. We can use integral (111) to express coefficients anyway, but it will be wasting the effort since the outcame is given to be zero.

So let us do the math for b_n and also for a_0:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} 1 \cdot dt = \frac{1}{\pi}[t]_0^{\pi} = \frac{\pi}{\pi} = 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin nt\, dt = \frac{1}{\pi}\int_0^{\pi} 1 \cdot \sin nt\, dt = \frac{1}{\pi}\left[-\frac{1}{n}\cos nt\right]_0^{\pi} =$$

$$= -\frac{1}{\pi n}[\cos nt]_0^{\pi} = \frac{-1}{\pi n}(\cos n\pi - \cos 0) = \frac{1-\cos n\pi}{\pi n} = \frac{1-(-1)^n}{\pi n}$$

Since $n = 1, 2, 3, \ldots$, the term $\cos n\pi$ could be simplified as the below table suggest to $(-1)^n$.

n	0	1	2	3
$\cos n\pi$	1	-1	1	-1

The coefficients are ready so the Fourier series can be evaluated from (110):

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n} \sin nt = \frac{1}{2} + \sum_{n=1}^{\text{odd}} \frac{2 \sin nt}{\pi n}$$

$$n = 0: \quad f(t) = \frac{a_0}{2} = \frac{1}{2}$$

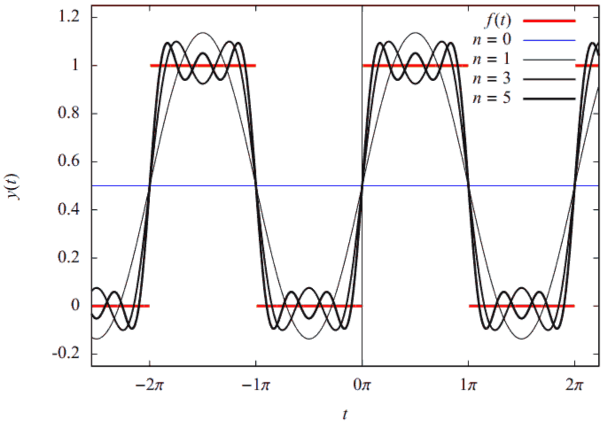
$$n = 1: \quad f(t) = \frac{1}{2} + \frac{1+1}{\pi} \sin t$$

$$n = 2: \quad f(t) = \frac{1}{2} + \frac{2}{\pi} \sin t + 0$$

$$n = 3: \quad f(t) = \frac{1}{2} + \frac{2}{\pi} \sin t + 0 + \frac{2}{3\pi} \sin 3t$$

$$n = 4: \quad f(t) = \frac{1}{2} + \frac{2}{\pi} \sin t + 0 + \frac{2}{3\pi} \sin 3t + 0$$

$$n = 5: \quad f(t) = \frac{1}{2} + \frac{2}{\pi} \sin t + 0 + \frac{2}{3\pi} \sin 3t + 0 + \frac{2}{5\pi} \sin 5t$$



The given function is a square function in red color.
The function labeled with $n = 0$ includes only the zeroest term $a_0/2$.
The function labeled with $n = 1$ includes $a_0/2$ and the first member.
The function labeled with $n = 3$ includes $a_0/2$ and the first three members.
The function labeled with $n = 5$ includes $a_0/2$ and the first five members.

Example

Find the Fourier series of function $x(t) = t^2$ periodic on interval $(-\pi, \pi)$

The given function is depicted in red within the below graph.

The function is symetric so it is going to consist of cosine terms, all $b_n = 0$. We are going to use integration by parts $\int u'v = uv - \int uv'$ to evaluate a_n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \cdot t^2\, dt = \frac{2}{\pi}\int_{0}^{\pi} t^2\, dt = \frac{2}{\pi}\left[\frac{t^3}{3}\right]_{t=0}^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt\, t^2\, dt = \frac{2}{\pi}\int_{0}^{\pi} \cos nt\, t^2\, dt = \left| \begin{matrix} u' = \cos nt & u = \frac{1}{n}\sin nt \\ v = t^2 & v' = 2t \end{matrix} \right| =$$

$$= \left[\frac{2}{\pi}\left(\frac{1}{n}\sin nt\, t^2\right)\right]_{t=0}^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{1}{n}\sin nt \cdot 2t\, dt =$$

$$= \frac{2}{n\pi}\left(\sin n\pi\, \pi^2 - 0\right) - \frac{4}{n\pi}\int_0^{\pi}\sin nt \cdot t\, dt = \left| \begin{matrix} u' = \sin nt & u = -\frac{1}{n}\cos nt \\ v = t & v' = 1 \end{matrix} \right| =$$

$$= 0 - \frac{4}{n\pi}\left(\left[-t\,\frac{1}{n}\cos nt\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\cos nt\, dt\right) =$$

$$= -\frac{4}{\pi}\left(\left(-\pi\,\frac{1}{n}\cos n\pi - 0\right) + \frac{1}{n}\left[\frac{1}{n}\sin nt\right]_0^{\pi}\right) = -\frac{4}{n\pi}\left(-\pi\,\frac{1}{n}\cos n\pi\right) = \frac{4}{n^2}\cos n\pi =$$

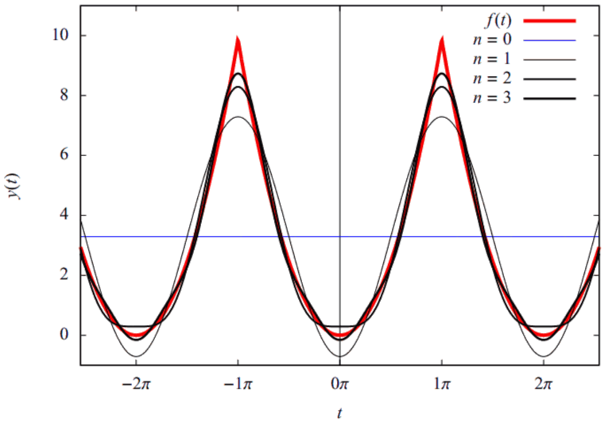
$$= \frac{4}{n^2}(-1)^n$$

Since $n = 1, 2, 3, \ldots$, the term $\cos n\pi$ could be simplified as the below table suggest to $(-1)^n$.

n	0	1	2	3
$\cos n\pi$	1	-1	1	-1

The coefficients are ready so the Fourier series can be evaluated from (110):

$$f(t) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nt$$



The given function is periodic t^2 function in red color.
The function labeled with $n = 0$ includes only the zeroest term $a_0/2$.
The function labeled with $n = 1$ includes $a_0/2$ and the first member.
The function labeled with $n = 2$ includes $a_0/2$ and the first two members.
The function labeled with $n = 3$ includes $a_0/2$ and the first three members.

Fourier serier

Solving DE by Fourier series

The **Fourier series decomposes periodic or bounded function into simple sinusoids.** It is difficult to work with functions as e.g. *square waves*, *sawtooth* are and it is easy to work with sines.

Modal analysis, natural frequencies, vibrations, dynamic behaviour

Fourier series are useful to study resonances of a system. Fourier analysis is able to decompose $f(t)$ into pure oscillations

$$f(t) = b_1 \sin \omega_1 t + b_1 \sin \omega_2 t + b_1 \sin \omega_3 t + \dots$$

That is very useful, because then we can analyze which frequencies are main or important within response. We can work with arbitrary periodic function (sine, cosine), solve individually and then recombine to obtain the solution of the original problem (superposition principle).

Natural frequencies characterize the basic behaviour of the system and indicate how the structure will respond to a dynamic loading. There are many reasons to compute the natural frequencies and mode shapes of the structure. For example a rotating machine is to be installed on the floor and it is necessary to determine if the operating frequency of the engine is close to one of the natural frequencies of the building. If the frequencies are too close the consequence might be structural damage or failure.

Example

Solve DE describing **free undamped motion** of spring mass system:

$$my'' + ky = 0 \qquad y(0) = 0,\ y(T) = 0$$

Note: the example is not solved using the Fourier series and serves to review spring-mass system and natural frequency.

We are solving free motion (no external force) of the mass on the spring. The boundary conditions tell us that at $t = 0$ and at $t = T$ the mass has to be found in position of equilibrium. If we use **natural angular frequency of the system $\omega_0 = \sqrt{k/m}$**, the same DE can be written as

$$y'' + \omega^2 y = 0.$$

Such DE is linear with constant coefficients and can be solved by means of auxiliary equation

$$m^2 + \omega^2 = 0 \implies m = \{-i\omega,\ i\omega\}$$

Then the solution is

$$y = C_1 e^{-i\omega t} + C_2 e^{i\omega t} = c_1 \sin \omega t + c_2 \cos \omega t.$$

Let us use boundary conditions to find $c_1,\ c_2$:

$$\begin{aligned} y(t=0) = 0: &\qquad 0 = c_1 \sin(\omega \cdot 0) + c_2 \cos(\omega \cdot 0) \implies c_2 = 0 \\ y(t=T) = 0: &\qquad 0 = c_1 \sin \omega T \end{aligned}$$

Now if we accept also $c_1 = 0$, we have trivial solution. Such solution is not much useful, because there is no motion described. The mass stays conserved at equilibrium position (does not move). We are looking for nontrivial solution, which comes from the fact that $\sin n\pi$ is zero for each $n = 1,\ 2,\ 3,\ \dots$

$$\omega t = n\pi \implies \omega = \frac{n\pi}{T}$$

In other words we can expres other (nontrivial) solution (regardless the value of c_1):

$$y = \sin\left(\frac{n\pi}{T}\, t\right).$$

The above solution can be listed as a sequence of solutions:

$$y_1 = \sin\frac{\pi t}{T}\,,\; y_2 = \sin\frac{2\pi t}{T}\,,\; y_3 = \sin\frac{3\pi t}{T}\,,\;\ldots$$

All these solutions are in chord with the complementary function found and at the same time satisfy given boundary conditions. The only issue is that the natural frequency of the system has to fit into given boundary conditions. The half-period of natural frequency has to be $T,\ 2T,\ 3T,\ \ldots$ and then the mass passes the requested position once, twice, three times, ... Otherwise the DE has only the trivial solution.

The numbers $\omega_n^2 = (n^2\pi^2)/T^2$, for which the problem posseses nontrivial solutions, are known as eigenvalues and the nontrivial solutions $y_1,\ y_2\ \ldots,\ y_n$ are known as eigenfunctions.

Note: $y'' + 5y = 0,\ y(0) = 0,\ y(T) = 0$ has no trivial solution. The mass is not going to appear back into requested zero position at the requested time T, because its natural frequency is not tuned with the time period T.

Example

Find the particular solution of the spring/mass system $y'' + \omega_0^2 y = f(t)$, if driving force of period $T = 2$ s is

$$f(t) = \begin{cases} 1 & 0 \le t < 1, \\ 0 & 1 \le t < 2. \end{cases}$$

and natural frequency of the system is $\omega_0 = 10$.

Firstly, let us discuss why we are finding particular solution y_p and not complementary function y_c. The complementary function is a *transient* solution: self motion with no driving force involved. In general, because of damping, $\lim_{t\to\infty} y_c(t) = 0$. We are rather interested into **the particular solution (steady periodic solution), which is the response to the driving force $f(t)$.**

Given periodic wave $f(t)$ expressed as a Fourier series is (not solved here)

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\text{odd}} \frac{\sin n t\pi}{n}\,.$$

So it is a series of $\sin n t\pi$ where each sine has a coefficient $b_n = 2/(\pi n)$ if $n = 1,\ 3,\ 5,\ \ldots$ **If we can solve this DE with/for one sine, then the same applies to the series of sines.** The nonhomogeneous function $f(t)$ on the right side involves sine. According to the method of undetermined coefficients we have to expect particular solution in the form $y_p = A\cos n t\pi + B\sin n t\pi$. But because there is no y' within DE, cosine terms have no origin to come from and can be overlooked:

$$y_p = \sum_{n=1}^{\text{odd}} B_n \sin n t\pi + A \qquad (116)$$

$$y_p'' = -\sum_{n=1}^{\text{odd}} B_n n^2\pi^2 \sin n t\pi$$

Let us do what we are used to do: substitute $y_p,\ y_p''$ into the DE $y'' + \omega_0^2 y = f(t)$.

$$-\sum_{n=1}^{\text{odd}} B_n n^2\pi^2 \sin n t\pi + \omega_0^2 \sum_{n=1}^{\text{odd}} B_n \sin n t\pi + \omega_0^2 A = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\text{odd}} \frac{\sin n t\pi}{n}$$

By method of undetermined coefficients, the coefficient A is easy to solve: $A = 1/2\omega_0^2$. The coefficient B_n can be solved in the same way as there is only B (with no series):

$$\sum_{n=1}^{\text{odd}} B_n(\omega_0^2 - n^2\pi^2)\sin n t\pi = \frac{2}{\pi}\sum_{n=1}^{\text{odd}} \frac{\sin n t\pi}{n}$$

$$B_n = \frac{2}{\pi}\cdot\frac{1}{n(\omega_0^2 - n^2\pi^2)}$$

The sine series of the solution is determined and let us list a few terms to investigate dominant frequencies.

$$n=1: \quad B_1 = \frac{2}{\pi}\frac{1}{1(10^2-1\pi^2)} = 0.007063$$
$$n=3: \quad B_3 = \frac{2}{\pi}\frac{1}{3(10^2-3^2\pi^2)} = 0.018992$$
$$n=5: \quad B_5 = \frac{2}{\pi}\frac{1}{5(10^2-5^2\pi^2)} = -0.000868$$
$$n=7: \quad B_7 = \frac{2}{\pi}\frac{1}{7(10^2-7^2\pi^2)} = -0.000237$$
$$n=9: \quad B_9 = \frac{2}{\pi}\frac{1}{9(10^2-9^2\pi^2)} = -0.000101$$

Now the found coefficients can be substituted back into particular solution (116).

$$y_p(t) = \frac{1}{2\omega_0^2} + 0.007063\sin\pi t + 0.018992\sin 3\pi t - 0.000868\sin 5\pi t - \\ - 0.000237\sin 7\pi t - 0.000101\sin 9\pi t - \dots$$

The amplitude of $\sin 3\pi t$ (i.e. for $\omega = 3\pi$) is the highest. **The system is not going to respond equally to all frequencies but favors frequencies close to its natural frequency.**

Example

Solve the spring/mass system $y'' + 0.05y' + 10.01y = f(t)$ if $f(t)$ is periodic function

$$f(t) = \begin{cases} 1 & 0 \le t < \pi, \\ -1 & \pi \le t < 2\pi. \end{cases}$$

From the given DE can be observed that it is a system with damping, where the mass is 1 kg and spring constant is 10.01 N/m. The natural frequency of the system is $\omega_0 = \sqrt{k/m} = \sqrt{10.01/1} = 3.164\ \mathrm{s}^{-1}$

Given periodic wave $f(t)$ expressed as a Fourier series is (not solved here)

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\text{odd}}\frac{1}{n}\sin nt.$$

So now we have DE with sine on the right side and we are looking for a particular solution y_p, which is a response to the driving force $f_t(t)$. We are not going to study complementary function y_c, which is only a solution of self motion, which is going—due to the damping—to zero anyway and is only a transient solution.

We have differential equation of the second order with constant coefficients with sine term on the right side. Such DE is easily solved by method of undetermined coefficients: let us collect expected terms $\sin nt$ and $\cos nt$ and then we have to solve constants A, B:

$$y_p = \sum_{n=1}^{\text{odd}} B_n \sin nt + \sum_{n=1}^{\text{odd}} A_n \cos nt \tag{117}$$
$$y_p' = \sum_{n=1}^{\text{odd}} B_n \cos nt \cdot n - \sum_{n=1}^{\text{odd}} A_n \sin nt \cdot n$$
$$y_p'' = -\sum_{n=1}^{\text{odd}} B_n \sin nt \cdot n^2 - \sum_{n=1}^{\text{odd}} A_n \cos nt \cdot n^2$$

Let us substitute y_p, y_p', y_p'' into DE and solve coefficients A, B by method of undetermined coefficients as usually:

$$
\begin{aligned}
&-\sum_{n=1}^{\text{odd}} B_n \sin nt \cdot n^2 - \sum_{n=1}^{\text{odd}} A_n \cos nt \cdot n^2 + \\
&+0.05\left(\sum_{n=1}^{\text{odd}} B_n \cos nt \cdot n - \sum_{n=1}^{\text{odd}} A_n \sin nt \cdot n\right) + \\
&+10.01\left(\sum_{n=1}^{\text{odd}} B_n \sin nt + \sum_{n=1}^{\text{odd}} A_n \cos nt\right) = \frac{4}{\pi}\sum_{n=1}^{\text{odd}} \frac{1}{n}\sin nt \\
&\sum_{n=1}^{\text{odd}} \sin nt(-B_n n^2 - 0.05A_n n + 10.01B_n) + \\
&\sum_{n=1}^{\text{odd}} \cos nt(-A_n n^2 + 0.05B_n n + 10.01A_n) = \frac{4}{\pi}\sum_{n=1}^{\text{odd}} \frac{1}{n}\sin nt
\end{aligned}
$$

$$-B_n n^2 - 0.05A_n n + 10.01B_n = \frac{4}{\pi n} \tag{118}$$

$$-A_n n^2 + 0.05B_n n + 10.01A_n = 0 \tag{119}$$

From (119) express A_n:

$$
\begin{aligned}
A_n(-n^2 + 10.01) &= -0.05B_n n \\
A_n &= \frac{-0.05B_n n}{10.01 - n^2} = \frac{0.05B_n n}{n^2 - 10.01}
\end{aligned}
$$

and substitute into (118):

$$
\begin{aligned}
-B_n n^2 - 0.05\,\frac{0.05B_n n}{n^2 - 10.01}\cdot n + 10.01B_n &= \frac{4}{\pi n} \\
&\vdots \\
B_n &= \frac{4(10.01 - n^2)}{\pi n\left(0.05^2 n^2 + (n^2 - 10.01)^2\right)} \\
A_n &= \frac{-0.2}{\pi\left(0.05^2 n^2 + (n^2 - 10.01)^2\right)}
\end{aligned}
$$

We can form the Fourier series of the response $y_p(t)$ if we use coefficients within (117). It is likely to be a messy motion, but this Fourier analysis served to decompose the motion into pure oscillations. The important information can be obtained by studying amplitudes C_n of particular frequencies:

$$
\begin{aligned}
n = 1: &\quad \left.\begin{aligned} A_1 &= -0.0008 \\ B_1 &= +0.1413 \end{aligned}\right\} \implies C_1 = 0.1413 \\
n = 3: &\quad \left.\begin{aligned} A_3 &= -0.0611 \\ B_3 &= +0.4111 \end{aligned}\right\} \implies C_3 = 0.4156 \\
n = 5: &\quad \left.\begin{aligned} A_5 &= -0.0003 \\ B_5 &= -0.0170 \end{aligned}\right\} \implies C_5 = 0.0170 \\
n = 7: &\quad \left.\begin{aligned} A_7 &= -0.0000 \\ B_7 &= -0.0047 \end{aligned}\right\} \implies C_7 = 0.0047
\end{aligned}
$$

The first and the second frequencies are the most important. For $n = 3$ is $\omega = 3$ and such frequency is close to the natural frequency $\omega_0 = 3.164\ \text{s}^{-1}$ of the system.

Note: since $\cos nt$ is phase delayed $\pi/2 = 90^\circ$ after $\sin nt$, there is a right angle between the actual amplitudes of sine and cosine. Therefore, the resultant (maximum amplitude) at particular frequency is $C_n = \sqrt{A_n^2 + B_n^2}$.

Example

Solve the spring/mass system $2x'' + 18\pi^2 x = f(t)$ if $f(t)$ is periodic function

$$f(t) = \begin{cases} -1 & -1 \le t < 0, \\ 1 & 1 \le t < 2. \end{cases}$$

From the given DE can be observed that it is a system with no damping, where the mass is likely 2 kg and the natural frequency of the system is $\omega_0 = \sqrt{k/m} = \sqrt{18\pi^2/2} = 3\pi$.

From auxiliary equation

$$\begin{aligned} m^2 + \omega^2 = 0 \implies m = \pm i\omega \\ y_c = C_1 e^{-i\omega x} + C_2 e^{i\omega x} \qquad \text{or} \\ y_c = c_1 \sin \omega x + c_2 \cos \omega x \end{aligned} \tag{120}$$

The complementary function was obtained from associated homogeneous DE and describes self motion with no external load involved. Since we are solving induced motion, y_c can be considered as a transient solution only.

Given periodic wave $f(t)$ expressed as a Fourier series is

$$f(t) = \sum_{n=1}^{\text{odd}} \frac{4}{n\pi} \sin nt\pi.$$

The function given has half-period $p = 1$. The series is odd, so only $b_n \cos \omega_n t$ terms are involved. We do not need to solve a_n. If we solve them anyway, they are going to turn to be zero.

$$\begin{aligned} b_n &= \frac{1}{p} \int_{-p}^{p} f(t) \sin nt \frac{\pi}{p} \; dt = \frac{2}{p} \int_0^p \sin nt\pi \, dt = \frac{2}{p} \left[-\cos nt\pi \frac{1}{n\pi} \right]_0^p = \\ &= \frac{2}{p} \left(-\frac{1}{n\pi} \right) (\cos np\pi - \cos 0) = -\frac{2}{p} \cdot \frac{1}{n\pi} (\cos n\pi - 1) = -\frac{2}{n\pi} (-2) \\ b_n &= \frac{4}{n\pi} \qquad \text{for } n \text{ odd} \end{aligned}$$

If the right side is $4/n\pi \sum \sin nt\pi$ and there is no x' within DE, the particular solution is expected in the form $x_p = B_n \sum \sin nt\pi$ for n odd, no cosine terms involved. **The trouble here is that sumation includes also $\sin 3t\pi$ and that term is already part of the complementary function (120) with arbitrary constant c_1.** As was discussed in chapter Undetermined coefficients—Superposition approach, we have to solve the particular solution for $t \cdot \sin 3t\pi$ and its derivatives.

The particular solution is then expected as

$$x_p(t) = x_{p1}(t) + x_{p_2}(t) \tag{121}$$

$$\begin{aligned} x_{p1}(t) &= \sum_{n=1, n\neq 3}^{\text{odd}} B_n \sin nt\pi \\ x_{p2}(t) &= A_3 \cos 3t\pi \cdot t + B_3 \sin 3t\pi \cdot t + C \cos 3t\pi + D \sin 3t\pi \end{aligned} \tag{122}$$

The terms $\cos 3t\pi$, $\sin 3t\pi$ are already included within y_c so they are voided from x_p again.

$$\begin{aligned} x_{p1}(t) &= \sum_{n=1, n\neq 3}^{\text{odd}} B_n \sin nt\pi \\ x''_{p1}(t) &= -\sum_{n=1, n\neq 3}^{\text{odd}} B_n \sin nt\pi \cdot n^2\pi^2 \end{aligned}$$

Let us substitute solution x_{p_1} into DE to solve the coefficients.

$$
\begin{aligned}
-2\sum B_n \sin nt\pi \cdot n^2\pi^2 + 18\pi^2 \sum B_n \sin nt\pi &= \sum \frac{4}{n\pi} \sin nt\pi \\
\sum \left(-2B_n n^2\pi^2 + 18\pi^2 B_n\right) \sin nt\pi &= \sum \frac{4}{n\pi} \\
-2B_n n^2\pi^2 + 18\pi^2 B_n &= \frac{4}{n\pi} \\
B_n(-2n^2\pi^2 + 18\pi^2) &= \frac{4}{n\pi} \\
B_n = \frac{4}{-2n^3\pi^3 + 18n\pi^3} &= \frac{2}{n\pi^3(9-n^2)}
\end{aligned}
$$

$$
\begin{aligned}
x_{p2}(t) &= A_3 \cos 3\pi t \cdot t + B_3 \sin 3\pi t \cdot t \\
x'_{p2}(t) &= A_3 \cos 3\pi t - 3\pi A_3 \sin 3\pi t \cdot t + B_3 \sin 3\pi t + 3\pi B_3 \cos 3\pi t \cdot t \\
x''_{p2}(t) &= -6\pi A_3 \sin 3\pi t - 9A_3\pi^2 \cos 3\pi t \cdot t + 6\pi B_3 \cos 3\pi t - 9B_3\pi^2 \sin 3\pi t \cdot t
\end{aligned}
$$

Let us substitute x_{p2} into DE $2x'' + 18\pi^2 x = f(t)$.

$$
\begin{aligned}
-12\pi A_3 \sin 3\pi t - 18A_3\pi^2 \cos 3\pi t \cdot t + 12\pi B_3 \cos 3\pi t - 18B_3\pi^2 \sin 3\pi t \cdot t +& \\
+18\pi^2 A_3 \cos 3\pi t \cdot t + 18\pi^2 B_3 \sin 3\pi t \cdot t &= \frac{4}{3\pi} \sin 3\pi t \\
-12\pi A_3 \sin 3\pi t + 12\pi B_3 \cos 3\pi t &= \frac{4}{3\pi} \sin 3\pi t
\end{aligned}
$$

And method of undetermined coefficients gives us

$$
\begin{aligned}
-12\pi A_3 &= \frac{4}{3\pi} \implies A_3 = -\frac{1}{9\pi^2} \\
B_3 &= 0
\end{aligned}
$$

Now we can go back to (121) and complete the particular solution.

$$
x_p(t) = -\frac{1}{9\pi^2} \cos 3\pi t + \sum_{n=1, n\neq 3}^{\text{odd}} \frac{2}{n\pi^3(9-n^2)} \sin nt\pi
$$

To study resonancy and which frequencies are main when the system is loaded by $f(t)$, we have to evaluate the coefficients:

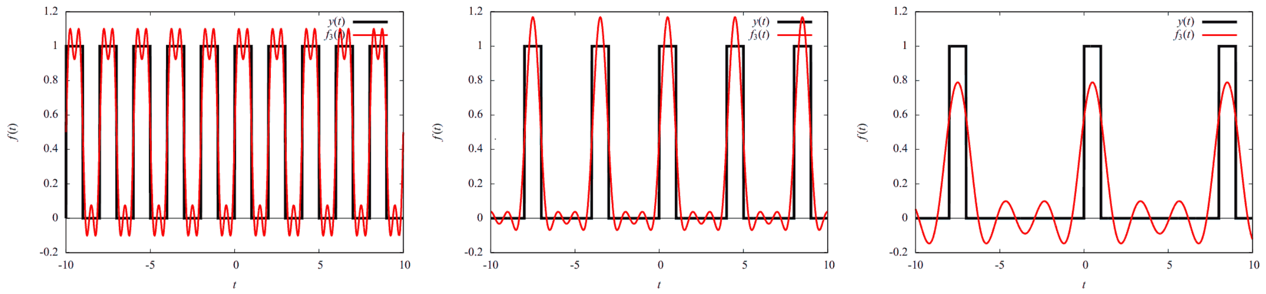
$$
\begin{aligned}
\omega_1 &= \pi : & B_1 &= 8.06 \times 10^{-3} \\
\omega_3 &= 3\pi : & A_3 &= -11.25 \times 10^{-3} \\
\omega_5 &= 5\pi : & B_5 &= -0.81 \times 10^{-3} \\
\omega_7 &= 7\pi : & B_7 &= -0.23 \times 10^{-3} \\
\omega_9 &= 9\pi : & B_9 &= -0.10 \times 10^{-3} \\
\omega_{11} &= 11\pi : & B_{11} &= -0.05 \times 10^{-3}
\end{aligned}
$$

Fourier series

Period T_0 stretched to infinity: envelope function

The Fourier series expects that the function $f(t)$ is periodic. What if we want to use Fourier series for a nonperiodic function? Either we can make such function periodic—but that is not always welcome—or we can make that function periodic with size of the period $T_0 \to \infty$. **If we stretch the period T_0 to infinity, then *periodic* expression become nonperiodic.**

So let us use Fourier series to represent periodic function and then observe what happens to Fourier series if we stretch its period T_0 to infinity.



Approximation of the square function by three terms of Fourier series ($n = 3$).
A—The square wave is of width 1, the period $T_0 = 2$
B—The square wave is of width 1, stretched to the period $T_0 = 4$
C—The square wave is of width 1, stretched to the period $T_0 = 8$

The three series depicted above use only the constant term $a_0/2$ and then first three terms b_1, b_2, b_3 for sine and a_1, a_2, a_3 for cosine functions. The coefficients for series were derived from integrals (not solved here) as:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt\,\frac{\pi}{p} + b_n \sin nt\,\frac{\pi}{p}\,, \qquad p = T_0/2$$

$$a_n = \frac{1}{\pi n} \sin \frac{\pi n}{p}$$

$$b_n = -\frac{1}{\pi n}\left(\cos \frac{\pi n}{p} - 1\right)$$

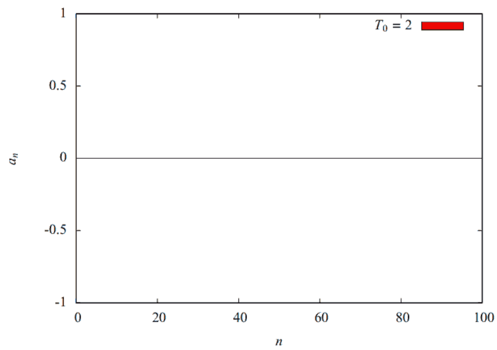
and if we list some of the first terms:

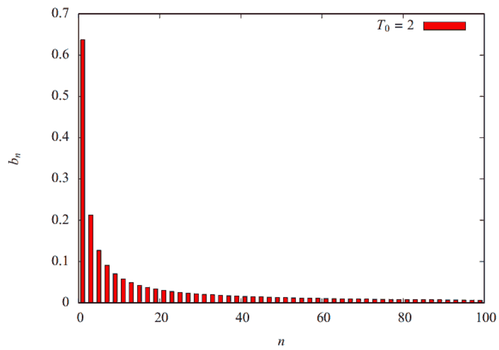
$$f_2(t) = \frac{1}{2} + 0.637 \sin \pi t + 0.212 \sin 3\pi t + 0.127 \sin 5\pi t + \ldots \qquad (123)$$

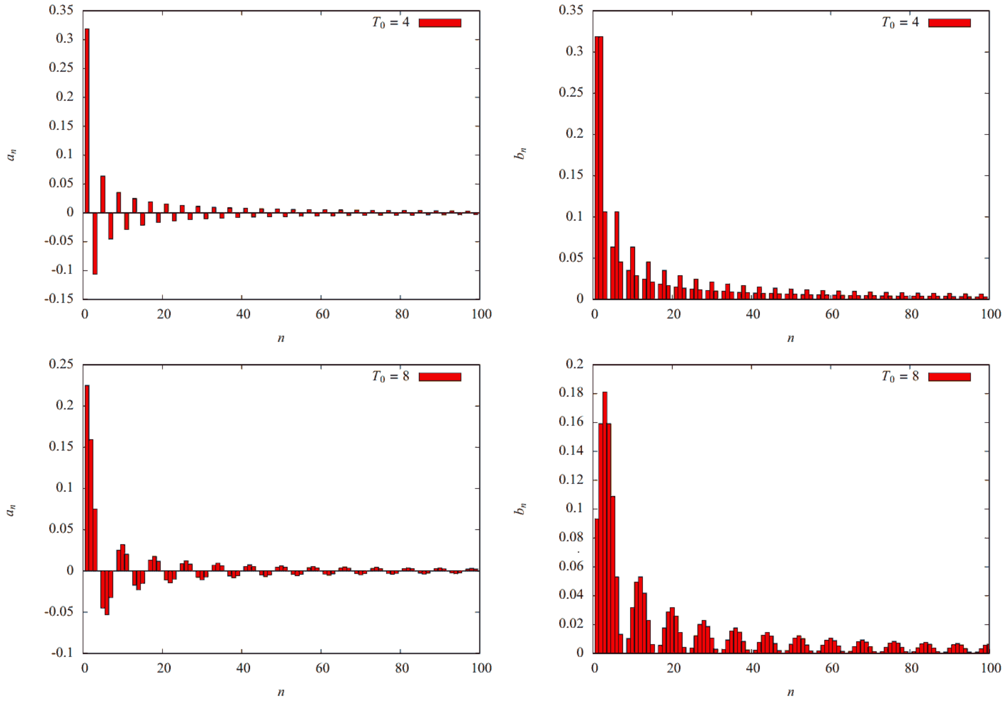
$$f_4(t) = \frac{1}{4} + 0.318 \sin \frac{\pi t}{2} + 0.318 \cos \frac{\pi t}{2} + 0.318 \sin \frac{2\pi t}{2} + 0.106 \sin \frac{3\pi t}{2} - 0.106 \cos \frac{3\pi t}{2} + \ldots \qquad (124)$$

$$f_8(t) = \frac{1}{8} + 0.093 \sin \frac{\pi t}{4} + 0.225 \cos \frac{\pi t}{4} + 0.159 \sin \frac{2\pi t}{4} + 0.159 \cos \frac{2\pi t}{4} + 0.181 \sin \frac{3\pi t}{4} + \ldots \qquad (125)$$

We can express and use more terms of these functions. The coefficient a_n, b_n were evaluated to build the series and first one hundred of them are presented in graphs below. All the coefficients a_n, b_n from (123) to (125) can be found in graphs below. It can be also read form the below graphs that the Fourier expansion of the above square wave function A ($T_0 = 2$) does not contain any cosine member $a_n \cos \omega_n t$:

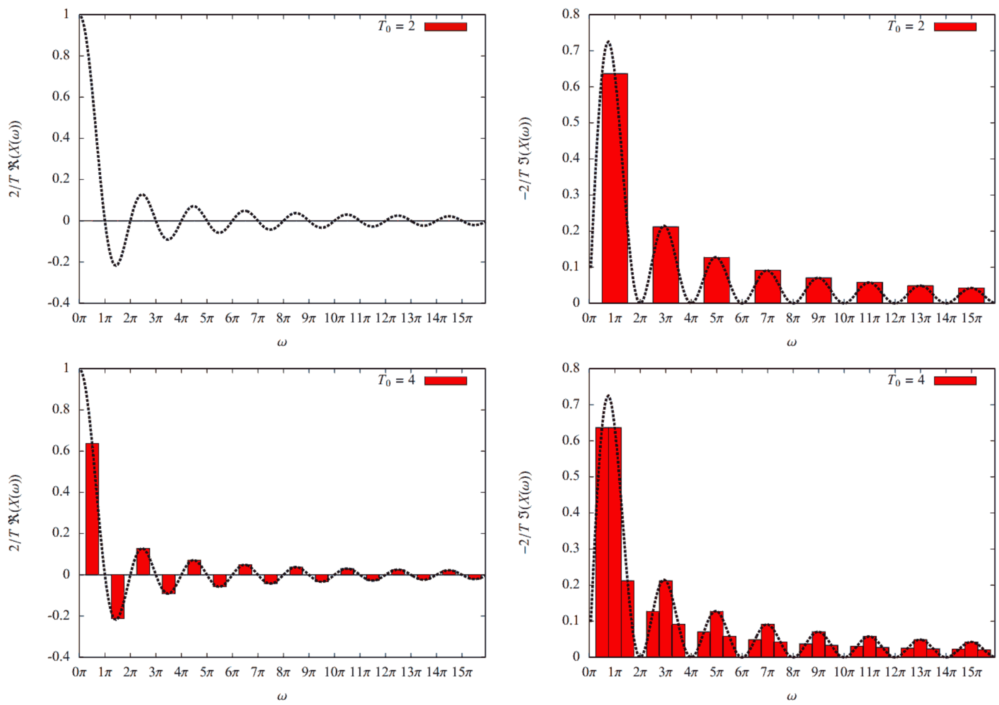


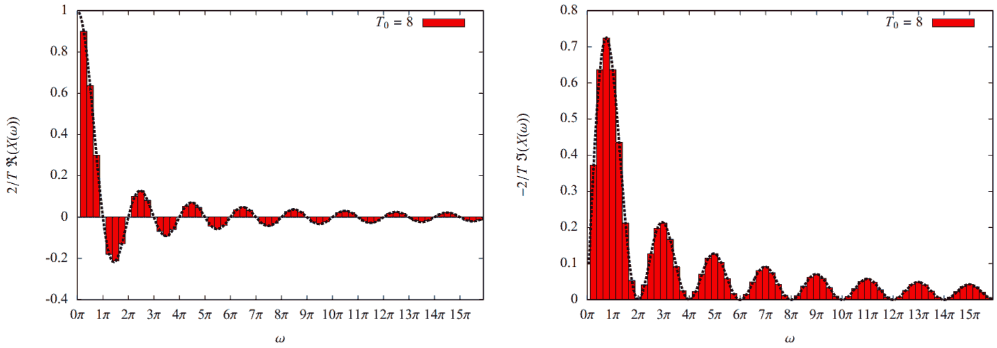




First one hundred coefficients $a_1, a_2, \ldots, a_{100}$ of cosines (on the left) and $b_1, b_2, \ldots, b_{100}$ for sines (on the right).
Although it might be not obvious, there is a repeating pattern for a_n and other pattern for b_n
1—The first row depicts coefficients a_n, b_n for the square wave A of width 1 with the period $T_0 = 2$
2—The second row depicts coefficients a_n, b_n for the square wave B of width 1 with period $T_0 = 4$
3—The third row depicts coefficients a_n, b_n for the square wave C of width 1 with period $T_0 = 8$

Now let us show the same graphs again with slight difference. The domain of horizontal axis will not be n (sequence of the coefficient) but it will be *frequency* ω_n associated with the coefficient a_n or b_n. The vertical axis will be normalized as well: we will divide its values by $T_0/2$ (T_0 is the period of the given function).





Coefficients a_n (left) and b_n (right) for cosine and sine terms until $\omega = 16\pi$.
The repeating pattern for both a_n and b_n is now obvious. ***The envelope is Fourier transform of function $x(t)$ made periodic with period $T_0 = \infty$***
1—The first row depicts coefficients a_n, b_n for the above square wave A of width 1 with the period $T_0 = 2$
2—The second row depicts coefficients a_n, b_n for the above square wave B of width 1 with period $T_0 = 4$
3—The third row depicts coefficients a_n, b_n for the above square wave C of width 1 with period $T_0 = 8$

It can be observed that: when the coefficients a_n, b_n for a function $x(t)$ are drawn in domain of frequencies, they are closed by envelopes (one for sines, the other one for cosines). The envelope remains the same regardless the period of the given function. If we stretch the period to infinity, there will be no red bars for each coefficient, it becomes continuous, as the envelope is.

The opposite way is possible then: all the coefficients a_n, b_n for Fourier series of a function can be read from the envelope (which is the Fourier transform of $x(t)$). Only need them to refactor considering the required period T_0.

Fourier transform

Because Fourier series involves both sines and cosines, it is reasonable rather to work with Fourier series in terms of complex numbers instead of real numbers:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \qquad \omega_n = n\,\frac{2\pi}{T_0}$$

In such approach the series operates with both positive and negative frequencies ω. Euler established the relationship between the trigonometric functions and the complex exponential function: $e^{it} = \cos t + i \sin t$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-iwt}\, dt \tag{126}$$

The envelope of coefficients is the Fourier transform $X(\omega)$ of the function $x(t)$. The function $x(t)$ is transformed into domain of frequencies, where it is defined as $X(\omega)$. The function is the same but its representation is different. Unlike in Laplace transform, Fourier transform is easily intepretable: transformed function $X(\omega)$ holds Fourier series' coefficients c_n for frequencies ω_n. It can be also seen how much of any given frequencies are present in $x(t)$.

Then particular coefficient for particular frequency ω_n and a periodicity T_0 is

$$c_n = \frac{X(\omega_n)}{T_0}\,. \tag{127}$$

Since the Fourier transform holds representation of the given $f(t)$ in the domain of frequencies, then by means of the above formula for every ω_n the coefficient of the Fourier series can be asked to reconstruct function $x(t)$ (in term of the Fourier series—see the example below). Note that c_n is a complex number. Conversion has to be done between real and complex numbers: $a_n = 2\Re(c_n)$, $b_n = -2\Im(c_n)$. The inverse Fourier transform is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t}\, d\omega$$

Note: although not mentioned in the chapter *Laplace transform*, also inverse Laplace transform has integral similar to the above one as a way to evaluate inverse Laplace transform. Similarly to Laplace transform, if $X(\omega)$ is Fourier transform of $x(t)$, then we write

$$\mathscr{F}[x(t)] = X(\omega)$$
$$\mathscr{F}^{-1}[X(\omega)] = x(t)$$

and transform can be also used to solve differential equations.

Now, when we have a definition and a tool (126) to transform a nonperiodic signal into frequency domain, we can make some example.

Example

Find a Fourier tranform of Dirac impulse function $\delta(t)$ (a unit impulse at $t=0$).

We will use definition (126) to transform given function into frequency domain:

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-iwt}\,dt=\int_{-\infty}^{\infty} \delta(t)e^{-iwt}\,dt$$

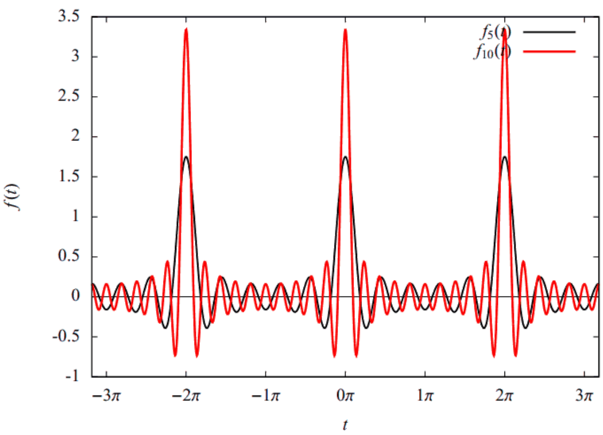
The value of $\delta(t)$ is zero everywhere except at $t=0$. And at $t=0$, when integrated, its area is one (from definition of Dirac delta function). Further, at the point $t=0$ the member $e^{-i\omega t}$ can be considered constant and taken out of the integral. Then $e^{-i\omega t}=e^{-i\omega\cdot 0}=1$:

$$X(\omega)=e^{-i\omega\cdot 0}\int_{-\infty}^{\infty} \delta(t)\,dt=1\cdot 1=1$$

The result says that all the frequencies have the same coefficient. We can draw the impulse as a Fourier series. Let us choose for the purpose $T_0=2\pi$. The coefficient from (127) $c_n=1/2\pi$ has real part only, so only cosine is involved with a coefficient $a_n=2\cdot\Re(c_n)=1/\pi$.

$$f(t)=\frac{1}{2\pi}+\sum\frac{1}{\pi}\cos\omega_n t=\frac{1}{2\pi}+\sum_{n=1}^{\infty}\frac{1}{\pi}\cos n\pi t$$

Let us draw how the impulse is represented by Fourier series if 5 and 10 terms are evaluated:



Dirac delta function $\delta(t)$ at $t=0$ expressed by cosines from Fourier transform in 5 and in 10 terms.

Note: there are many functions on $(-\infty,\infty)$, which do not posses a Fourier transform; e.g. $x(t)=t,\ x(t)=t^2,\ x(t)=e^t$.

Partial differential equation (PDE)

Problems from mathematics and physics

Differential equations describe many different physical systems, ranging form gravitation to fluid mechanics. They are difficult to study, they usually have individual equation, which needs to be analyzed as a separate problem.

A fundamental question for any **PDE** is the existence and uniqueness of a solution for given **boundary conditions.** For nonlinear equations these questions are in general hard to answer.

Heat equation

FIGURE 12.3.1 Temperatures in a rod of length L

*This problem occurs in the theory of heat flow—heat is transferred by conduction in a rod. The flow of heat occurs only in the x dimension (the surface along the length is insulated). Such problem might sound weird, but it is also the problem of **heat flow through the wall.***

The equation describing the heat equation problem is

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad k > 0.$$

- x is a point along the rod (a rather simply spatial dimension);
- t is time;
- k is a physical constant related to the material;
- $u(x, t)$ represents temperature at any point at any time.

So here is the equation describing the problem and **the problem is defined also by its boundary and initial values or conditions.** For example

- temperature at a point is fixed (typically at the beginning/end of the rod);
- distribution of temperature in the body at time $t = 0$ is given;
- flow at boundary is not allowed (the end is insulated): $\partial u / \partial x = 0|_{x=L}$.

Laplace's equation

Laplace's equation is useful for solving many physical problems such as electrostatic, gravitational or velocity in fluid mechanics. It can be also interpreted as a steady state temperature distribution. Laplace's equation in two and three dimensions is abbreviated as

$$\nabla^2 u = 0, \qquad \text{where}$$
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{(for 2D).}$$

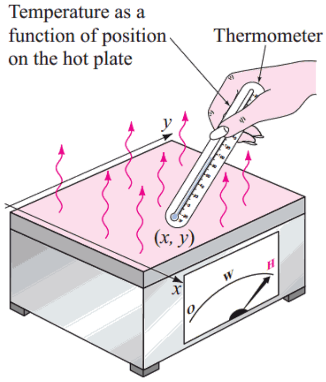


FIGURE 12.2.3 Steady-state temperatures in a rectangular plate

Steady state solution $u(x, y)$ of temperature distribution at any point (x, y) according to boundary conditions.

Partial differential equation (PDE)

These kinds of DE occur in problems involving temperature distribution, vibrations and potentials. These problems are described by relatively simple **linear second order PDEs.** The procedures described below are used to **reduce the problem into two or more linear DE.**

We are going to work with function u, which as a function of two variables: either $x(x, y)$ or $u(x, t)$. Then the second order PDE which is discussed can be described as

$$A\,\frac{\partial^2 u}{\partial x^2} + B\,\frac{\partial^2 u}{\partial x \partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + F(u) = G,$$

where the coefficients $A,\ B,\ C,\ \ldots,\ G$ are functions of x and y.

Separation of variables

We have no ambitions to find the general solution of such DE. That would be too difficult and it is not so useful. We are looking for particular solutions in a form

$$u(x, y) = X(x) \cdot Y(y)$$

Then it is sometimes possible to reduce given linear PDE (in two variables) into two ODEs. Assuming that the solution is $u(x, y) = X(x)Y(y)$ we have to be able to differentiate $u(x, y)$ with respect to x, y and so on:

$$\frac{\partial u}{x} = X'Y, \qquad \frac{\partial u}{y} = XY', \qquad \frac{\partial^2 u}{x^2} = X''Y, \qquad \frac{\partial^2 u}{y^2} = XY''.$$

The above statements come from a product rule (one function is multiplied by another): let us have for example $u(x, y) = XY = (x^2)(y^3)$. Then $\partial u/\partial x = (2x)(y^3) + (x^2)(0) = 2xy^3 = X'Y$.

If we can separate X to the left hand side of equation and Y to the right side of equation, then: the left side depends on x and what is on the right (in terms of x) is a constant. The right side depends on y on what is on the left (in terms of y) is constant. So we have two ODE which are solved against a constant (which is often called *separation constant* λ). It is easily demonstrated on examples:

Example

Use the method *separation of variables* to find product solution of PDE

$$\frac{\partial u}{\partial x} + 3\,\frac{\partial u}{\partial y} = 0$$

We are looking for a solution in the form $y(x, y) = X(x) \cdot Y(y)$. Let us use the solution within given PDE:

$$X'Y + 3XY' = 0 \qquad / \cdot \frac{1}{X}\,\frac{1}{Y}$$
$$X'\,\frac{1}{X} = -3Y'\,\frac{1}{Y}$$

The left side is a constant from the point of view of the right side and conversely. We can solve two ODE

$$X'\,\frac{1}{X} = \lambda, \qquad -3Y'\,\frac{1}{Y} = \lambda,$$

where λ is an arbitrary constant, X is $X(x)$, Y is $Y(y)$. Let us solve $X'\,\frac{1}{X} = \lambda$ first:

$$X'\,\frac{1}{X} = \lambda$$
$$X' - \lambda X = 0$$
$$m - \lambda = 0 \implies m = \lambda$$
$$X(x) = c_1 e^{\lambda x}$$

Now let us solve $Y(y)$:

$$
\begin{aligned}
-3Y'\frac{1}{Y} &= \lambda \\
-3Y' - \lambda Y &= 0 \\
3Y' + \lambda Y &= 0 \\
3m + \lambda = 0 &\implies m = -\frac{\lambda}{3} \\
Y(y) &= c_2 e^{-\lambda/3y}
\end{aligned}
$$

Since we expected solution in the form $u(x, y) = X(x)Y(y)$:

$$
u(x, y) = c_1 e^{\lambda x} \cdot c_2 e^{-\lambda/3y} = c_3 e^{\lambda(x-1/3y)}
$$

The constant λ was laid as an arbitrary constant also. But it would be weird to use λ, let us rather use c_4 instead:

$$
\underline{\underline{u(x, y) = c_3 e^{c_4(3x-y)}}}
$$

Example

Use the method *separation of variables* to find product solution of PDE $u_x = u_y + u$

We are looking for a solution in the form $y(x, y) = X(x) \cdot Y(y)$ of PDE

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} + u
$$

Let us use the solution $y(x, y) = X(x) \cdot Y(y)$ within given PDE:

$$
\begin{aligned}
X'Y &= Y'X + XY \qquad / \cdot \frac{1}{X}\frac{1}{Y} \\
\frac{1}{X}X' &= \frac{1}{Y}Y' + 1
\end{aligned}
$$

As in the previous example, we have to separate variable x from y. The consequence is we are solving two ODE (128) and (129) with arbitrary constant then λ instead of PDE:

$$
\frac{1}{X}X' = \lambda = \frac{1}{Y}Y' + 1
$$

$$
\frac{1}{X}X' = \lambda \tag{128}
$$

$$
\frac{1}{Y}' + 1 = \lambda \tag{129}
$$

The first DE $\frac{1}{X}X' = \lambda$ has been solved in the previous example and its solution is $X(x) = c_1 e^{\lambda x}$. The second one has solution

$$
\begin{aligned}
\frac{1}{Y}Y' + 1 &= \lambda \qquad / \cdot Y \\
1Y' + Y(1-\lambda) &= 0 \\
m + (1-\lambda) = 0 &\implies m = \lambda - 1 \\
Y(y) &= c_2 e^{(\lambda-1)y}
\end{aligned}
$$

Then we collect the solution of two ODE into expected form

$$
u(x, y) = X(x)Y(y) = c_1 e^{\lambda x} \cdot c_2 e^{(\lambda-1)y} = c_3 e^{\lambda x + \lambda y - y} = c_3 e^{\lambda(x+y)-y}
$$

Finally it is good idea to exchange λ to some other symbol:

$$
\underline{\underline{u(x, y) = c_3 e^{c_4(x+y)-y}}}.
$$

We might differentiate $u(x,y)$ to express $\partial u/\partial x$ and $\partial u/\partial y$ in order to substitute back into PDE. That will provide as a confidence that no mistake was made through calculations.

Example

Use the method *separation of variables* to find product solution of PDE

$$y\,\frac{\partial u}{\partial x} + x\,\frac{\partial u}{\partial y} = 0 \tag{130}$$

We are looking for a solution $u(x,y) = X(x)Y(y)$. So we have to insert expected solution into DE:

$$\begin{aligned} yX'Y + xXY' &= 0 \qquad / \cdot \frac{1}{X'}\,\frac{1}{Y'} \\ y\,\frac{1}{Y'}\,Y + xX\,\frac{1}{X'} &= 0 \\ y\,\frac{1}{Y'}\,Y &= -xX\,\frac{1}{X'} \end{aligned}$$

The variables are separated. On the left is function of x and whatever is now being left on the right side can be considered as constant. And conversely. So we can have two DE which are solved against arbitrary constant λ:

$$y\,\frac{1}{Y'}\,Y = \lambda$$

The above DE is ordinary differential equation and is solved here with changed notation (for convenience) from $Y(y)$ to $y(x)$.

$$\begin{aligned} x\,\frac{1}{y'}\,y &= \lambda \\ xy &= \lambda y' \\ y' - \frac{1}{\lambda}\,xy &= 0 \end{aligned}$$

The DE turned to be solvable as a linear DE with $P(x) = -1/\lambda\,x$. Such DE is solved by means of integrating factor $\mu(x) = e^{\int P(x)\,dx}$

$$\mu(x) = e^{-1/\lambda \int x\,dx} = e^{-\frac{1}{\lambda}\cdot\frac{x^2}{2}}$$

Whole DE is multiplied by the integrating factor and then we anticipate $d/dx\;\mu(x)y$ on the left hand side and such term is trivial to integrate:

$$\begin{aligned} \frac{d}{dx}\,\mu(x)y &= 0 \\ y' \cdot e^{-\frac{1}{\lambda}\cdot\frac{x^2}{2}} - \frac{1}{\lambda}\,xy \cdot e^{-\frac{1}{\lambda}\cdot\frac{x^2}{2}} &= 0 \\ \frac{d}{dx}\,e^{-\frac{1}{\lambda}\cdot\frac{x^2}{2}} \cdot y &= 0 \\ e^{-\frac{1}{\lambda}\cdot\frac{x^2}{2}} \cdot y &= c_1 \\ y(x) &= c_1 e^{\frac{1}{\lambda}\cdot\frac{x^2}{2}} \end{aligned}$$

Note: could be solved easily also by separating variables.

So we have $Y(y) = c_1 e^{\frac{1}{\lambda}\cdot\frac{y^2}{2}}$. That implies that the second solution is $X(x) = c_2 e^{-\frac{1}{\lambda}\cdot\frac{x^2}{2}}$. Then the solution $u(x,y)$ is

$$u(x,y) = X(x)\cdot Y(y) = c_1 e^{\frac{1}{\lambda}\cdot\frac{y^2}{2}} \cdot c_2 e^{-\frac{1}{\lambda}\cdot\frac{x^2}{2}} = c_3 e^{\frac{1}{2\lambda}(y^2 - x^2)}$$

After exchanging c_4 for $-1/2\lambda$

$$u(x,y) = c_3 e^{c_4(x^2-y^2)}$$

We might have a desire to check the solution for a correctness. In that case, let us compute $\partial u/\partial x$, $\partial u/\partial y$ and check whether equality holds when substituted into given DE (130).

$$\begin{aligned}
\frac{\partial u}{\partial x} &= c_3 \cdot 2x \cdot c_4 \cdot e^{c_4(x^2-y^2)} \\
\frac{\partial u}{\partial y} &= c_3 \cdot (-2y) \cdot c_4 \cdot e^{c_4(x^2-y^2)} \\
0 &= y \cdot c_3 \cdot 2x \cdot c_4 \cdot e^{c_4(x^2-y^2)} + x \cdot c_3 \cdot (-2y) \cdot c_4 \cdot e^{c_4(x^2-y^2)} \\
0 &= 0
\end{aligned}$$

The other way to check for correctness is to use numerical check by calculator. Let us choose randomly some numbers for arbitrary constants and for x, y as well. Let us evaluate numericaly the slope from the solution. When all substituted into DE, the equality have to (roughly) hold.

$$\begin{aligned}
&c_3 = 2,\ c_4 = 3,\ x = 1,\ y = 2 \\
&u(x=1, y=2) = 2.468196 \times 10^{-4} = u \\
&u(x=1.001, y=2) = 2.483057 \times 10^{-4} = u_{\Delta x} \\
&u(x=1, y=2.001) = 2.438747 \times 10^{-4} = u_{\Delta y} \\
&\frac{\partial u}{\partial x} = \frac{\Delta u}{\Delta x} = \frac{u_{\Delta x} - u}{0.001} = 1.4861 \times 10^{-3} \\
&\frac{\partial u}{\partial y} = \frac{\Delta u}{\Delta y} = \frac{u_{\Delta y} - u}{0.001} = -2.9449 \times 10^{-3}
\end{aligned}$$

Now substitute prepared values into given DE (130):

$$2 \cdot 1.4861 \times 10^{-3} + 1 \cdot (-2.9449 \times 10^{-3}) = 0.0273 \times 10^{-3}$$

It can be considered that the equality holds. We did not expect that we are going to get exactly to zero.

Example

Use separation of variables to find solution of

$$k\frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}, \qquad k > 0.$$

We are interested into solution $u(x,y) = X(x)T(t)$:

$$\begin{aligned}
k \cdot X''T - XT &= XT' \qquad / \cdot \frac{1}{X} \cdot \frac{1}{T} \\
k \cdot X''\frac{1}{X} - 1 &= T'\frac{1}{T} \\
kX''\frac{1}{X} &= 1 + T'\frac{1}{T} \qquad (131)
\end{aligned}$$

The variables are separated. Now we can solve two ODE against separation constant λ.

The left hand side from (131) is solved against constant λ:

$$\begin{aligned}
kX''\frac{1}{X} &= \lambda \\
kX'' &= \lambda X \\
kX'' - \lambda X &= 0
\end{aligned}$$

Now it depends on the configuration of the problem (on the value of λ) what the solution would be.

$\boldsymbol{\lambda = 0:}$	$\boldsymbol{\lambda > 0\ (\lambda = \alpha^2 k):}$	$\boldsymbol{\lambda < 0\ (\lambda = -\alpha^2 k):}$
$kX'' - 0 = 0$ $X'' = 0$ $X(x) = c_1 x + c_2$	$kX'' - \alpha^2 k = 0$ $X'' - \alpha^2 = 0 \implies$ $\implies m = \{\alpha, -\alpha\}$ $X(x) = c_3 \sinh \alpha x +$ $+ c_4 \cosh \alpha x$	$kX'' + \alpha^2 k = 0$ $X'' + \alpha^2 = 0 \implies$ $\implies m = \{i\alpha, -i\alpha\}$ $X(x) = c_5 \sin \alpha x +$ $+ c_6 \cos \alpha x$

Note: the above solution with $\sinh \alpha x$, $\cosh \alpha x$ is equivalent to $C_3 e^{-\alpha x} + C_4 e^{\alpha x}$, because $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.

The right hand side from (131) is solved against constant λ:

$$\begin{aligned} 1 + T' \frac{1}{T} &= \lambda \\ T' - (\lambda - 1)T &= 0 \\ T' + (1 - \lambda)T &= 0 \\ m + (1 - \lambda) = 0 &\implies m = \lambda - 1 \\ T(t) &= c_7 e^{(\lambda - 1)t} \end{aligned}$$

Finally, let us combine X and T into $u(x, y)$:

1. When $\lambda = 0$:
   $\underline{\underline{u(x, y) = (c_1 + c_2 x)e^{-t}}}$
2. When $\lambda > 0\ (\lambda = \alpha^2 k)$:
   $\underline{\underline{u(x, y)}} = c_7 e^{(\lambda-1)t} \cdot (c_3 \sinh \alpha x + c_4 \cosh \alpha x) = \underline{\underline{e^{-t} e^{\alpha^2 kt} \cdot c_7 (c_3 \sinh \alpha x + c_4 \cosh \alpha x)}}$
3. When $\lambda < 0\ (\lambda = -\alpha^2 k)$:
   $\underline{\underline{u(x, y)}} = c_7 e^{(\lambda-1)t} \cdot (c_5 \sin \alpha x + c_6 \cos \alpha x) = \underline{\underline{e^{-t} e^{-\alpha^2 kt} \cdot c_7 (c_5 \sin \alpha x + c_6 \cos \alpha x)}}$

Example

Solve the heat equation with given initial and boundary conditions:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad \begin{aligned} u(x, 0) &= f(x) \\ u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned}$$

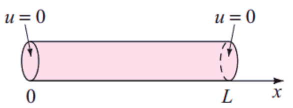


FIGURE 12.3.1 Temperatures in a rod of length L

We have an equation describing physical problem and we are given initial conditions (distribution of temperature at $t = 0$ is described by the function $f(x)$) and boundary conditions (temperature at the start and the end of the rod is set to zero. Well setting (or keeping) the temperature to zero is somehow unrealistic. But simple.

We are finding solution $u(x, t) = X(x)T(t)$. Let us observe boundary condition substituted into the solution:

- $u(0, t) = 0 \implies T(t)X(0) = 0$
- $u(L, t) = 0 \implies T(t)X(L) = 0$

That means either $T(t)$ has to be zero or $X(0)$ and $X(L)$ has to be zero. The first one, i.e. $T(t) = 0$ is not reasonable to expect. Because the time is factor of the problem. If $T(t) = 0$, then the solution $u(x, y) = X(x) \cdot T(t) = 0$ can not be useful nor valid. If $T(t)$ is not zero, then we are looking for

$X(L) = X(0) = 0$:

$$\begin{aligned} XT' &= kX''T \\ T'\frac{1}{Tk} &= X''\frac{1}{X} \end{aligned} \tag{132}$$

We are solving $X''\frac{1}{X} = \lambda$ from (132).

$$\begin{aligned} X'' &= \lambda X \\ X'' - \lambda X &= 0 \end{aligned}$$

Let us replace α^2 for λ, then the expressions and statements are clearer.

1. $\lambda = 0$:

$$X'' = 0 \implies X(x) = c_1 x + c_2$$

2. $\lambda < 0\ (\lambda = -\alpha^2)$:

$$X'' + \alpha^2 = 0 \implies X(x) = c_1 \sin \alpha x + c_2 \cos \alpha x \tag{133}$$

3. $\lambda > 0\ (\lambda = \alpha^2)$:

$$X'' - \alpha^2 = 0 \implies X(x) = c_1 \sinh \alpha x + c_2 \cosh \alpha x$$

All those trivial solutions with $c_1 = c_2 = 0$ are really solutions for given initial values but are not useful and do not solve given problem. Our hope is focused on $\sin \alpha x$ in (133). Because if we apply $X(0) = 0$ and $X(L) = 0$ into solution (133) we are able to get **nontrivial solution:**

$$\begin{aligned} \alpha x &= \pi n \\ \alpha L &= \pi n \implies \alpha = \frac{\pi n}{L} \\ X(x) &= c_1 \sin \frac{\pi n}{L} x \end{aligned}$$

We are solving the left hand side of (132):

$$\begin{aligned} T'\frac{1}{Tk} &= \lambda \\ T' &= \lambda Tk \\ T' - \lambda Tk &= 0 \\ m - \lambda k &= 0 \implies m = \lambda k \\ T &= c_3 e^{\lambda kt} \end{aligned} \tag{134}$$

So we have a **sequence of solutions** for $n = 1, 2, 3, \ldots$ combined from (133) and (134)

$$u_n(x,t) = T(t)X(x) = A_n e^{-\frac{\pi^2 n^2}{L^2}kt} \cdot \sin \frac{\pi n}{L} x$$

Now at time $t = 0$ inital conditions are given: $u(x, t = 0) = f(x)$. So let us investigate the solution there:

$$u_n(x, t = 0) = f(x) = A_n \sin \frac{\pi n}{L} x$$

Our solution does not look well. Because if $f(x)$ is *some* function, how the equality could hold? Is $f(x)$ restricted to be only a sine wave? So we have to admit that u_n is not a solution. But we know from the superposition rule, that if u_1 and u_2 is a solution, then $c_1 u_1 + c_2 u_2$ is solution as well. Therefore we can write

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{\pi^2 n^2}{L^2}kt} \cdot \sin \frac{\pi n}{L} x$$

Let us set condition at $t = 0$ again:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{\pi n}{L} x$$

That is good enough. That is what we recognize as a Fourier series. Only instead of coefficient b_n we have A_n. So the function is now described by Fourier series and we can express its coefficients from known formula $b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin nx \frac{\pi}{p} \, dx$:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin nx \frac{\pi}{L} \, dx$$

Then the solution is

$$\underline{\underline{u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(x) \sin nx \frac{\pi}{L} \, dx \right) \cdot e^{-\frac{\pi^2 n^2}{L^2} kt} \cdot \sin \frac{\pi n}{L} x}}$$